Technical Mathematics, 2nd Edition

TECHNICAL MATHEMATICS, 2ND EDITION

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Bob Brown

Contents

Introduction

This developmental-level mathematics textbook is intended for career-technical students. It was made possible by grants from Open Oregon Educational Resources, which supports the development and implementation of high-quality materials at low or no cost to community college and university students.

Changes from the first edition

I won't list every detail, but here are the broad strokes.

- Added Modules 31, 32, 33, 34.
- Revised real-world data and examples from 2019 that felt outdated.
- Deleted or revised complicated explanations and examples.
- Redrew some low-resolution geometry figures.
- Cleaned up the formatting and made the online text easier to navigate.

To the student

This textbook is designed to bring a bit of fun to what can often be a boring or intimidating subject. Many of the examples are taken from everyday life rather than the trades, which helps make the material accessible to students in any field.

There are very few step-by-step examples, because—let's face it—does anyone ever read those? The general structure is a brief introduction of a topic followed by some practice exercises, a brief introduction of the next topic followed by more practice exercises, etc. Therefore, this book is not designed for self-study unless you already have some experience with the material and are simply looking for a refresher.

I have tried to indicate when a calculator would be appropriate, but your instructor has the final say on that. My personal philosophy is that a calculator is a useful tool, but the user needs to know enough to be able to estimate and judge whether a result is reasonable. I don't mention calculators until we get to formulas in Module 7 because I want students to be able to work with fractions, decimals, negatives, exponents, and all that stuff without a calculator if necessary.

In most careers, nobody is going to say you cheated if you use technology such as a calculator or spreadsheet; the important thing is that you use the available technology to get the correct answer. However, you're in school trying to obtain a degree or certificate, which means that you need to demonstrate certain knowledge and skills. Therefore, you need to show that you can (1) do the calculations without technology if necessary, and (2) determine whether an answer given by technology is reasonable or not.

One final thought about learning in general. After going through a particular topic, let's say fractions, you may feel that you understand everything, but there is a big difference between recognizing something when it's shown to you and actually doing it yourself. Think about learning to drive a car; your driving class has a lot of reading and videos and lectures to familiarize you with what you need to know, but it's impossible to learn to drive a car without actually driving the car yourself. Reading about it and watching other people do it is not enough; it is crucial that you actually *practice the techniques yourself* to be sure that you have mastered them. (And then you need to continue practicing the techniques to maintain that mastery, but that's another speech for another time.) Two ways to know that you've really and truly learned something are to either perform it yourself with no guidance or explain it to someone else. If you can follow someone else's explanation of how to add fractions, you're partway there. If you can explain how to add fractions to someone else, though? Now you're in business!

I wish you the best of luck, and I hope you enjoy pictures of cats!

To the instructor

To paraphrase the key points from above:

Many of the examples are taken from everyday life rather than specific trades. At my school, students in this course come from many different CTE programs—welding, machining, automotive, landscaping, horticulture, renewable energy—but not other programs such as nursing, electrical, or wastewater. I also have a few students in other non-CTE programs who don't need a course in the math of CNC machining. You should of course feel free to supplement the textbook with topics you'd like to emphasize… Or heck, this is an OER with a CC-BY-NC-SA 4.0 license; you can copy, revise, and remix it however you like! The only restrictions are that you must credit the author where you use his work and you may not charge money for your product, excluding a reasonable printing cost for a print edition. If you have a Pressbooks account, you can clone the book and then start revising as needed.

There are very few step-by-step worked out examples in the textbook. I believe that they add clutter without adding value. I provide my students with a playlist of YouTube videos that I ask them to watch before class, because watching and listening to someone work a problem—with the ability to pause, rewind, replay—is generally more effective than reading a static example printed on the page. Rather than reading examples that I've typed out step by step, students can actively try the exercises for themselves, with guidance from you.

If you use MyOpenMath for homework assignments, I have good news! I have written and/or collected over 600 questions and organized them into assignments for each module. You can copy course #**181716** and use it as you please.

I shared my personal philosophy on calculators above, but you're the expert on your situation and what your students need. Similarly, you (or your department) will decide how students will be assessed in your course. Will students be allowed to use the textbook or their notes during exams? Will you provide formulas and other reference information? Will a calculator be allowed for some or all of each exam? Will you be giving exams or assigning some kind of mastery worksheets instead? This paragraph is getting beyond the scope of a textbook introduction, but I would ask that you think about what it's important for your students to know how to do, with and without technology, and how they can best demonstrate that knowledge.

To everyone

Whether you are a student or instructor, I would love to hear your thoughts on the book and whether it works well for you. Feel free to let me know about any errors you find or suggestions for improvements. In fact, the trigonometry modules were written at the request of—and in collaboration with—instructors who reached out to let me know what they needed. I had intended to include a module on arc length but ran out of time, energy, and ideas.

Special thanks go out to Bob Brown at Kishwaukee College, who provided many of the diagrams and exercises in Modules 31 & 32.

Extra special thanks go out to my Math 50 students at CCC who have given me first-hand evidence of what worked well in the first edition and what did not. 😅

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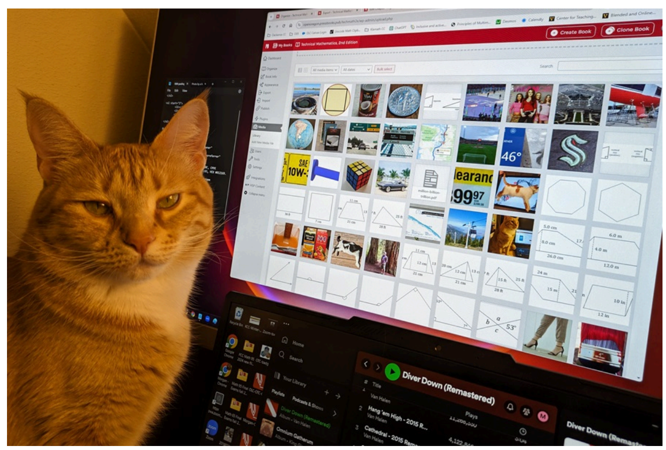


Pepper the cat oversees the revisions.

[1]

Order of Operations

Photo by bastamanography on flickr.

We'll begin with a look at order of operations. In many situations in life, the order in which we perform certain actions is important. For example, if you are putting on your shoes and socks, you need to put a sock on your foot before you put a shoe on that foot. However, if you put on your left sock first, it doesn't really matter whether the next thing you put on is your left shoe or your right sock, as long as you don't try putting on your right shoe next. A multi-step math calculation can be the same way; you might have some steps that need to happen in a specific sequence, but there may be some steps that you can do in whatever order you prefer.

Evaluating an Expression

To ***evaluate*** an expression means to simplify it and find its value.

Exercises

1. Evaluate by performing the addition first: $12 - 2 + 3$
2. Evaluate by performing the subtraction first: $12 - 2 + 3$

When we evaluate an expression, we want to have a single correct answer. It isn't very helpful for the answer to be "maybe 7, or maybe 13". Mathematicians have decided on an order of operations, which tells us which steps should be done before other steps. You can think of them as the rules of the road.

Order of Operations: PEMDAS

P: Work inside of **parentheses** or grouping symbols, following the order PEMDAS as necessary inside the grouping symbols.

E: Evaluate **exponents**.

MD: Perform **multiplications** and **divisions** from left to right.

AS: Perform **additions** and **subtractions** from left to right.

Exercises

Evaluate each expression.

3. $12 - (2 + 3)$
4. $12 - 2 + 3$

Based on Exercises 3 & 4, we can see that Exercise 1 told us to use the wrong order of operations. If there are no parentheses, we must evaluate $12 - 2 + 3$ by *first* performing the subtraction and *then* performing the addition.

Before we move on, you should be aware that there are a handful of ways to show multiplication. All of the following represent 3×4:

$$3 \cdot 4 \qquad 3 * 4 \qquad 3(4) \qquad (3)4 \qquad (3)(4)$$

In this textbook, you will most often see the dot, like $3 \cdot 4$, or parentheses directly next to a number, like $3(4)$. We tend to avoid using the 3×4 symbol because it can be mistaken for the letter x.

Exercises

Evaluate each expression.

5. $12 \div (3 \cdot 2)$

6. $12 \div 3 \cdot 2$

7. $5(1 + 3) - 2$

8. $5(1) + (3 - 2)$

Exponents

An exponent indicates repeated multiplication. For example, $6^2 = 6 \cdot 6 = 36$ and $4^3 = 4 \cdot 4 \cdot 4 = 64$. The exponent tells us how many factors of the base are being multiplied together.

Exercises

Evaluate each expression.

9. $3^2 + 4^2$

10. $(3 + 4)^2$

11. $(7 + 3)(7 - 5)^3$

12. $7 + 3(7 - 5)^3$

Grouping Symbols

In the next set of exercises, the only differences are the parentheses, but every exercise has a different answer.

Exercises

Evaluate each expression.

13. $39 - 7 \cdot 2 + 3$

14. $(39 - 7) \cdot 2 + 3$

15. $39 - (7 \cdot 2 + 3)$

16. $39 - 7 \cdot (2 + 3)$

17. $(39 - 7) \cdot (2 + 3)$

It is possible to have grouping symbols nested within grouping symbols; for example, $7 + (5^2 - (3(17 - 12 \div 4) + 2 \cdot 5) \div 4)$.

To make it somewhat easier to match up the pairs of left and right parentheses, we can use square brackets instead: $7 + (5^2 - [3(17 - 12 \div 4) + 2 \cdot 5] \div 4)$.

Exercises

Simplify the expression.

18. $7 + (5^2 - [3(17 - 12 \div 4) + 2 \cdot 5] \div 4)$

A fraction bar is another grouping symbol; it tells us to perform all of the steps on the top and separately perform all of the steps down below. The final step is to divide the top number by the bottom number.

Exercises

Evaluate each expression.

19. $\frac{15-1}{6+1}$

20. $\frac{(7+2)\cdot 4}{18\div(3+3)}$

21. $\frac{5\cdot 4^2}{2}$

22. $\frac{(5\cdot 4)^2}{2}$

23. $\frac{(5-1)^2}{2+6}$

24. $(5-1)^2 \div 2 + 6$

We will look at formulas in a later module, but let's finish by translating from words to a mathematical expression.

Exercises

25. You can find the approximate Fahrenheit temperature by doubling the Celsius temperature and adding 30. If the temperature is $9°$C, what is the approximate Fahrenheit temperature? Write an expression and evaluate it.

26. You can find the approximate Celsius temperature by subtracting 30 from the Fahrenheit temperature and then dividing by 2. If the temperature is $72°$F, what is the approximate Celsius temperature? Write an expression and evaluate it.

Exercise Answers

[2]

Negative Numbers

Negative numbers are a fact of life, from winter temperatures to our bank accounts. (And occasionally elevators, if they go underground.)

Before we start calculating with negative numbers, we'll take a look at ***absolute value***. This will make it easier for us to talk about what we're doing when we add, subtract, multiply, or divide signed numbers.

Absolute Value

The absolute value of a number is its distance from 0. You can think of it as the size of a number without identifying it as positive or negative. Numbers with the same absolute value but different signs, such as 3 and -3, are called ***opposites***. The absolute value of -3 is 3, and the absolute value of 3 is also 3, because both numbers are 3 units away from 0.

We use a pair of straight vertical bars to indicate absolute value; for example, $|-3| = 3$ and $|3| = 3$.

Exercises

Evaluate each expression.

1. $|-5|$

2. $|5|$

Adding Negative Numbers

To add two negative numbers, add their absolute values (i.e., ignore the negative signs) and make the final answer negative.

Exercises

Perform each addition.

3. $-8+(-7)$

4. $-13+(-9)$

To add a positive number and a negative number, we *subtract* the smaller absolute value from the larger. If the positive number has the larger absolute value, the final answer is positive. If the negative number has the larger absolute value, the final answer is negative.

Exercises

Perform each addition.

5. $7+(-3)$

6. $-7+3$

7. $14 + (-23)$

8. $-14 + 23$

9. The temperature at noon on a chilly Monday was -7°F. By the next day at noon, the temperature had risen 25°F. What was the temperature at noon on Tuesday?

If an expression consists of only additions, we can break the rules for order of operations and add the numbers in whatever order we choose.

Exercises

Evaluate each expression using any shortcuts that you notice.

10. $-10 + 4 + (-4) + 3 + 10$

11. $-291 + 73 + (-9) + 27$

Subtracting Negative Numbers

The following image shows part of a paystub in which an $\$18$ payment needed to be made, but the payroll folks wanted to track the payment in the deductions category. Of course, a positive number in the deductions will subtract money away from the paycheck. Here, though, a deduction of negative 18 dollars has the effect of *adding* 18 dollars to the paycheck. Subtracting a negative amount is equivalent to adding a positive amount.

Deductions after Federal Tax	
Faculty Union Dues	$27.00
Stipend for Part Time Faculty	-$18.00
Workers Compensation Hourly Assessment	$0.13

To subtract two signed numbers, we *add* the first number to the *opposite* of the second number.

Exercises

Perform each subtraction.

12. $5 - 2$

13. $2 - 5$

14. $-2 - 5$

15. $-5 - 2$

16. $2 - (-5)$

17. $5 - (-2)$

18. $-2 - (-5)$

19. $-5 - (-2)$

Absolute Value, Revisited

Absolute value can be useful when we want to find the difference between two numbers but we want the result to be positive. For example, suppose that the temperature in Portland, Oregon is $43°$F, and the temperature in Portland, Maine is $-12°$F. What is the difference in temperature? The simplest way to find the difference is to do $43 - (-12) = 43 + 12 = 55$, and you would report that as a difference of "fifty-five degrees Fahrenheit". If you instead did $-12 - 43 = -55$, it would sound a bit strange to say the the difference is "negative fifty-five degrees Fahrenheit" and you would most likely ignore the negative sign when reporting the difference. To guarantee that the result of a subtraction is positive, we can put absolute value bars around the entire calculation. This is sometimes called the ***positive difference***.

Exercises

Evaluate each expression.

20. $|-12 - 43|$

21. $|43 - (-12)|$

22. The lowest point in Colorado is on the Arikaree River, with an elevation $3,317$ feet above sea level. The highest point in Colorado is the peak of Mount Elbert, with an elevation $14,440$ feet above sea level.[1] Find the positive difference between these elevations.

23. The lowest point in Louisiana is in New Orleans, with an elevation 8 feet below sea level. The highest point in Louisiana is the peak of Driskill Mountain, with an elevation 535 feet above sea level.[2] Find the positive difference between these elevations.

Multiplying Negative Numbers

Suppose you spend 3 dollars on a coffee every day. We could represent spending 3 dollars as a negative number, -3 dollars. Over the course of a 5-day work week, you would spend 15 dollars, which we could represent as -15 dollars. This shows that $-3 \cdot 5 = -15$, or $5 \cdot -3 = -15$.

If two numbers with *opposite* signs are multiplied, the product is negative.

Exercises

Find each product.

24. $-4 \cdot 3$

25. $5(-8)$

Going back to our coffee example, we saw that $5(-3) = -15$. Therefore, the *opposite* of $5(-3)$ must be positive 15. Because -5 is the opposite of 5, this implies that $-5(-3) = 15$.

If two numbers with the *same* sign are multiplied, the product is positive.

WARNING! These rules are different from the rules for addition; be careful not to mix them up.

Exercises

Find each product.

26. $-2(-9)$

27. $-3(-7)$

Recall that an exponent represents a repeated multiplication. Let's see what happens when we raise a negative number to an exponent.

Exercises

Evaluate each expression.

28. $(-2)^2$

29. $(-2)^3$

30. $(-2)^4$

31. $(-2)^5$

If a negative number is raised to an *even* power, the result is positive.
If a negative number is raised to an *odd* power, the result is negative.

Dividing Negative Numbers

Let's go back to the coffee example we saw earlier: $-3 \cdot 5 = -15$. We can rewrite this fact using division and see that $-15 \div 5 = -3$; a negative divided by a positive gives a negative result. Also, $-15 \div -3 = 5$; a negative divided by a negative gives a positive result. This means that the rules for division work exactly like the rules for multiplication.

If two numbers with *opposite* signs are divided, the quotient is negative.
If two numbers with the *same* sign are divided, the quotient is positive.

Exercises

Find each quotient.

32. $-42 \div 6$

33. $32 \div (-8)$

34. $-27 \div (-3)$

35. $0 \div 4$

36. $0 \div (-4)$

37. $4 \div 0$

Go ahead and check those last three exercises with a calculator. Any surprises?

> 0 divided by another number is 0.
>
> A number divided by 0 is *undefined*, or *not a real number*.

Here's a quick explanation of why $4 \div 0$ can't be a real number. Suppose that there is a mystery number, which we'll call n, such that $4 \div 0 = n$. Then we can rewrite this division as a related multiplication, $n \cdot 0 = 4$. But because 0 times any number is 0, the left side of this equation is 0, and we get the result that $0 = 4$, which doesn't make sense. Therefore, there is no such number n, and $4 \div 0$ cannot be a real number.

Order of Operations with Negative Numbers

P: Work inside of **parentheses** or grouping symbols, following the order PEMDAS as necessary.

E: Evaluate **exponents**.

MD: Perform **multiplications** and **divisions** from left to right.

AS: Perform **additions** and **subtractions** from left to right.

Exercises

Evaluate each expression using the order of operations.

38. $(2-5)^2 \cdot 2+1$

39. $2-5^2 \cdot (2+1)$

40. $[7(-2)+16] \div 2$

41. $7(-2)+16 \div 2$

42. $\frac{1-3^4}{2(5)}$

43. $\frac{(1-3)^4}{2} \cdot 5$

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/Colorado
2. Source: https://en.wikipedia.org/wiki/Louisiana

[3]

Decimals

Photo by Christian Collins on flickr.

Decimal notation is based on powers of 10: 0.1 is one tenth, 0.01 is one hundredth, 0.001 is one thousandth, and so on.

thousands	hundreds	tens	ones/units	.	tenths	hundredths	thousandths

Exercises

Write each number.

1. ninety and twenty-three hundredths
2. seven and fifty-six thousandths

Adding & Subtracting Decimals

Before you add or subtract decimals, you must line up the decimal points.

Exercises

Add each pair of numbers.

3. $3.75 + 12.8$

4. $71.085 + 112.93$

When subtracting, you may need to add zeros to the first number so you can borrow correctly.

Exercises

Subtract each pair of numbers.

5. $46.57 - 38.29$

6. $82.78 - 67.024$

Multiplying Decimals

To multiply decimal numbers:

1. Temporarily ignore the decimal points.
2. Multiply the numbers as though they are whole numbers.
3. Add the total number of decimal digits in the two numbers you multiplied. The result will have that number of digits to the right of the decimal point.

Note: You do NOT need to line up the decimal points when you are multiplying.

Exercises

Multiply each pair of numbers.

7. $143 \cdot 29$

8. $143 \cdot 2.9$

9. $14.3 \cdot 2.9$

10. $1.43 \cdot 2.9$

11. $375 \cdot 175$

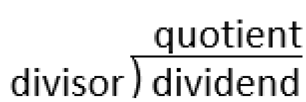
12. $375 \cdot 0.175$

13. $3.75 \cdot 1.75$

14. Evie worked 37.5 hours at a pay rate of $\$17.50$ per hour. How much did she earn in total?

Dividing Decimals

Let's review everyone's favorite topic, long division. The three parts of a division are named as follows: dividend ÷ divisor = quotient. When this is written with a long division symbol, the dividend is inside the symbol, the divisor is on the left, and the quotient is the answer we create on top.



To divide by a decimal:

1. Write in long division form.
2. Move the decimal point of the divisor until it is a whole number.
3. Move the decimal point of the dividend the same number of places to the right.
4. Place the decimal point in the quotient directly above the decimal point in the dividend.
5. Divide the numbers as though they are whole numbers.
6. If necessary, add zeros to the right of the last digit of the dividend to continue.

Exercises

Divide each pair of numbers.

15. $974 \div 4$
16. $974 \div 0.4$
17. $9,740 \div 0.04$
18. $0.0974 \div 0.004$

Rounding Numbers

It is often necessary to round a number to a specified place value. We will see more specific instructions in Modules 5 & 6, but let's review the basics of rounding a number.

Rounding a number:

1. Locate the **rounding digit** in the place to which you are rounding.
2. Look at the **test digit** directly to the right of the rounding digit.
3. If the test digit is 5 or greater, increase the rounding digit by 1 and drop all digits to its right.
4. If the test digit is less than 5, keep the rounding digit the same and drop all digits to its right.

Exercises

Round each number to the indicated place value.

19. $6,375$ (thousands)

20. $6,375$ (tens)

21. 0.7149 (hundredths)

22. 0.7149 (thousandths)

If a decimal answer goes on and on, it may be practical to round it off.

Exercises

23. A subscription to *The Chicago Manual of Style Online* costs $\$44.00$. Determine the monthly cost, rounded to the nearest cent.

24. In the summer of 1919, a military convoy (including Lt. Col. Dwight Eisenhower) drove from Washington, D.C. to San Francisco to assess the condition of the nation's developing highway system. The journal entry for August 1 says "Good dirt roads. Made 82 miles in 11 hrs."[1] What was the convoy's effective speed in miles per hour for that day? Round your result to the nearest tenth.

Exercise Answers

Notes

1. Source: https://after-ike.com/logbook-1919-transcontinental-military-convoy/. See https://www.nytimes.com/2019/07/07/opinion/the-most-important-road-trip-in-american-history.html if you're interested in the historical context.

[4]

Fractions

5 out of 4 people have trouble with fractions.

Working with fractions is one of the most hated/feared/avoided topics in lower-level mathematics. If you've always struggled with fractions, now is the time to face them. Don't avoid them and hope they'll go away. (They won't.) We'll start with the basics of what fractions are and proceed from there.

Writing Fractions

A fraction describes equal parts of a whole: $\frac{\text{part}}{\text{whole}}$

Using official math vocabulary: $\frac{\text{numerator}}{\text{denominator}}$

Exercises

The month of April had 11 rainy days and 19 days that were not rainy.

1. What fraction of the days were rainy?
2. What fraction of the days were not rainy?

Simplifying Fractions

Two fractions are ***equivalent*** if they represent the same number. (The same portion of a whole.) To build an equivalent fraction, multiply the numerator and denominator by the same number.

Exercises

3. Write $\frac{4}{5}$ as an equivalent fraction with a denominator of 15.

4. Write $\frac{2}{3}$ as an equivalent fraction with a denominator of 12.

Many fractions can be ***simplified***, or reduced. Here are four special cases.

Exercises

Simplify each fraction, if possible.

5. $\frac{7}{1}$

6. $\frac{7}{7}$

7. $\frac{0}{7}$

8. $\frac{7}{0}$

A fraction is ***completely reduced***, or in ***simplest form***, or in ***lowest terms***, when the numerator and denominator have no common factors other than 1. To reduce a fraction, divide the numerator and denominator by the same number.

Exercises

Reduce each fraction to simplest form.

9. $\frac{9}{12}$

10. $\frac{10}{6}$

Multiplying Fractions

To multiply fractions, multiply the numerators and multiply the denominators straight across. If possible, simplify your answer.

Exercises

Multiply each pair of numbers. Be sure that each answer is in simplest form.

11. $8 \cdot \frac{1}{4}$

12. $\frac{6}{7} \cdot \frac{7}{12}$

13. $\frac{5}{8} \cdot \frac{2}{3}$

14. $\frac{6}{5} \cdot \frac{10}{12}$

To find a fraction **of** a number, multiply.

Exercises

15. To pass his workplace training, Nathan must correctly answer at least $\frac{9}{10}$ of 50 questions. How many questions must he answer correctly to pass the training?

Dividing Fractions

To divide by a fraction, multiply by the reciprocal of the second number. (Flip the second fraction upside-down.)

Exercises

Divide. Be sure that each answer is in simplest form.

16. $12 \div \frac{3}{4}$

17. $\frac{3}{10} \div \frac{1}{2}$

18. Suppose you need to measure 2 cups of flour, but the only scoop you can find is $\frac{1}{3}$ cup. How many scoops of flour will you need?

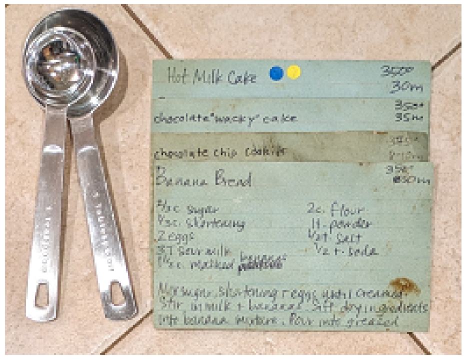
Comparing Fractions

If two fractions have the same denominator, we can simply compare their numerators.

If two fractions have different denominators, we can rewrite them with a common denominator and then compare their numerators.

Exercises

19. Banana bread recipe A requires $\frac{3}{4}$ cup of sugar, whereas banana bread recipe B requires $\frac{2}{3}$ cup of sugar. Which recipe requires more sugar?



Adding & Subtracting Fractions

To add or subtract two fractions with the same denominator, add or subtract the numerators and keep the common denominator.

Exercises

20. Jack ate $\frac{3}{8}$ of a pizza. Mack ate $\frac{1}{8}$ of the pizza. What fraction of the pizza did they eat together?

21. Tracy ate $\frac{5}{6}$ of a pizza. Stacy ate $\frac{1}{6}$ of the pizza. How much more of the pizza did Tracy eat?

To add or subtract two fractions with different denominators, first write them with a common denominator. Then add or subtract them.

Exercises

A $\frac{3}{8}$-inch thick sheet of plywood is going to be laid onto a $\frac{1}{4}$-inch thick sheet of plywood.

22. What is the combined thickness of the two sheets?

23. What is the difference in thickness of the two sheets of plywood?

Jacqueline budgets $\frac{1}{4}$ of her monthly income for food and $\frac{1}{3}$ of her monthly income for rent.

24. What fraction of her monthly income does she budget for these two expenses combined?

25. What fraction more of her monthly income does she budget for her rent than for her food?

Fractions and Decimals

To write a fraction as a decimal, divide the numerator by the denominator.

A decimal that ends (eventually has a remainder of 0) is called a terminating decimal. Fun fact: If the denominator of a fraction has no prime factors other than 2's and 5's, the decimal will terminate. Also, the fraction can be built up to have a denominator of 10, or 100, or $1,000$...

Exercises

Write each fraction as a decimal.

26. $\frac{11}{4}$

27. $\frac{7}{20}$

A decimal that continues a pattern of digits is called a repeating decimal. We can represent the repeating digits by using either an overbar or ellipsis (three dots)...

Exercises

Write each fraction as a decimal.

28. $\frac{5}{9}$

29. $\frac{18}{11}$

Improper Fractions & Mixed Numbers

A fraction which has a larger numerator than denominator is called an ***improper fraction***. Because an improper fraction is larger than 1, it can also be written as a ***mixed number***, with a whole number followed by a fractional part.

Keep in mind that a mixed number represents an addition, not a multiplication. For example, $6\frac{2}{3}$ means $6 + \frac{2}{3}$, not $6 \cdot \frac{2}{3}$.

To write an improper fraction as a mixed number:

1. Divide the numerator by the denominator to get the whole number part.
2. The remainder after dividing is the new numerator.
3. Keep the same denominator.

Exercises

Rewrite each improper fraction as a mixed number.

30. $\frac{23}{2}$

31. $\frac{14}{3}$

To write a mixed number as an improper fraction:

1. Multiply the whole number part by the denominator.
2. Add this result to the original numerator to get the new numerator.
3. Keep the same denominator.

Exercises

Rewrite each mixed number as an improper fraction.

32. $2\frac{1}{5}$

33. $6\frac{2}{3}$

Adding or subtracting mixed numbers can be fairly simple or more complicated, depending on the numbers. One approach is to work with the fractional parts separately from the whole numbers. For example, $5\frac{2}{3} - 3\frac{1}{2}$ can be rewritten as $5 + \frac{2}{3} + (-3) + (-\frac{1}{2})$ and rearranged to $[5 + (-3)] + [\frac{2}{3} + (-\frac{1}{2})]$. Then $5 + (-3) = 2$ and, with a little more work, $\frac{2}{3} + (-\frac{1}{2}) = \frac{1}{6}$, so the result is $2\frac{1}{6}$.

Exercises

34. $7\frac{5}{8} + 2\frac{3}{4}$

35. $7\frac{5}{8} - 2\frac{3}{4}$

Multiplying or dividing mixed numbers is more complicated than it may appear. Change any mixed numbers into improper fractions before doing the calculation, then change the answer back to a mixed number if possible.

Exercises

36. Multiply: $3\frac{1}{2} \cdot 2\frac{1}{3}$

37. $5\frac{1}{2}$ cups of water will be divided equally into 3 jars. How much water will go into each jar?

Exercise Answers

[5]

Accuracy and Significant Figures

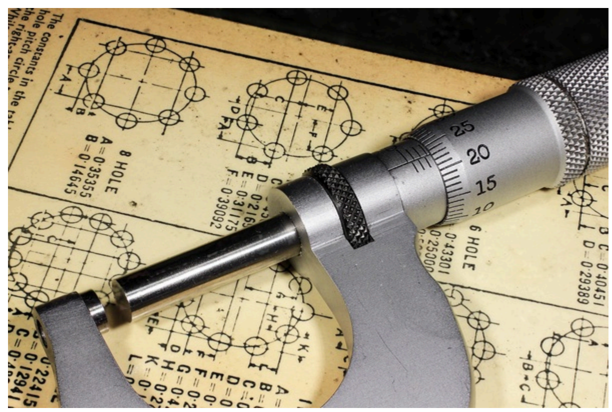


Photo by Tudor Barker on flickr.

In the first few modules, we rarely concerned ourselves with rounding; we assumed that every number we were told was exact and we didn't have to worry about any measurement error. However, every measurement contains some error. A standard sheet of paper is 8.5 inches wide and 11 inches high, but it's possible that the actual measurements could be closer to 8.4999 and 11.0001 inches. Even if we measure something very carefully, with very sensitive instruments, we should assume that there could be some small measurement error.[1]

Exact Values and Approximations

A number is an ***exact value*** if it is the result of counting or a definition.

A number is an ***approximation*** if it is the result of a measurement or of rounding.

Exercises

Identify each number as an exact value or an approximation.

1. An inch is $\frac{1}{12}$ of a foot.
2. This board is 78 inches long.
3. There are 14 students in class.
4. A car's tachometer reads $3,000$ rpm.
5. A right angle measures $90°$.
6. The angle of elevation of a ramp is $4°$.

Accuracy and Significant Figures

African-American women were vital to NASA's success in the 1960s, as shown in the movie *Hidden Figures*. Photo by NASA/Kim Shiflett on flickr.

Because measurements are inexact, we need to consider how accurate they are. This requires us to think about ***significant figures***—often abbreviated "sig figs" in conversation—which are the digits in the measurement that we trust to be correct. The ***accuracy*** of a number is equal to the number of significant figures. (By the way, the terms "significant digits" and "significant figures" are used interchangeably.) The following rules aren't particularly difficult to understand but they can take time to absorb and internalize, so we'll include lots of examples and exercises.

Significant Figures

1. All nonzero digits are significant.
   Ex: $12,345$ has five sig figs, and 123.45 has five sig figs.
2. All zeros between other nonzero digits are significant.
   Ex: $10,045$ has five sig figs, and 100.45 has five sig figs.
3. Any zeros to the right of a decimal number are significant.
   Ex: 123 has three sig figs, but 123.00 has five sig figs.
4. Zeros on the left of a decimal number are NOT significant.
   Ex: 0.123 has three sig figs, and 0.00123 has three sig figs.
5. Zeros on the right of a whole number are NOT significant unless they are marked with an overbar.
   Ex: $12,300$ has three sig figs, but $12,30\overline{0}$ has five sig figs.

Another way to think about #4 and #5 above is that zeros that are merely showing the place value—where the decimal point belongs—are NOT significant.

Exercises

Determine the accuracy (i.e., the number of significant figures) of each number.

7. $63,400$
8. $63,040$
9. $63,004$
10. 0.085
11. 0.0805
12. 0.08050

In 1856, the first official measurement of the height of Mount Everest—called *Sagarmatha* in Nepal and *Chomolungma* in Tibet—was announced. The height was determined to be exactly $29,000$ feet, but there was concern that people would think this was only a rough estimate rounded to the nearest thousand feet. Therefore, the height was announced as $29,002$ feet, so that everyone seeing that number would believe that the measurement was correct to the nearest foot.[2] Yes, to demonstrate the correctness of the measurement, an incorrect measurement was announced.

Mt. Everest, Lohtse, and Nupse in the early morning. Photo by Ralf Kayser on flickr.

Instead of fudging a number like $29,000$ to show that it is correct to the nearest foot, we can write it with an an ***overbar*** to indicate that the zeros are significant. Putting $29,00\overline{0}$ in a newspaper headline in 1856 would probably have confused people, but you can handle it because you're in a math class. Writing $29,00\overline{0}$ is our way of saying "Really, to the nearest foot, it's exactly $29,000$ feet!"

Exercises

Determine the accuracy (i.e., the number of significant figures) of each number.

13. $29,000$

14. $29,\overline{0}00$

15. $29,0\overline{0}0$

16. $29,00\overline{0}$

Two things to remember: we don't put an overbar over a nonzero digit, and we don't need an overbar for any zeros on the right of a decimal number because those are already understood to be significant.

Accuracy-Based Rounding

As we saw in Module 3, it is often necessary to round a number. We often round to a certain place value, such as the nearest hundredth, but there is another way to round. ***Accuracy-based rounding*** considers the number of significant figures rather than the place value.

Accuracy-based rounding:

1. Locate the **rounding digit** to which you are rounding by counting from the left until you have the correct number of significant figures.
2. Look at the **test digit** directly to the right of the rounding digit.
3. If the test digit is 5 or greater, increase the rounding digit by 1 and drop all digits to its right. If the test digit is less than 5, keep the rounding digit the same and drop all digits to its right.

Exercises

Round each number so that it has the indicated number of significant figures.

17. $51,837$ (three sig figs)

18. $51,837$ (four sig figs)

19. 4.2782 (two sig figs)

20. 4.2782 (three sig figs)

When the rounding digit of a whole number is a 9 that gets rounded up to a 0, we must write an overbar above that 0.

Similarly, when the rounding digit of a decimal number is a 9 that gets rounded up to a 0, we must include the 0 in that decimal place.

Exercises

Round each number so that it has the indicated number of significant figures. Be sure to include trailing zeros or an overbar if necessary.

21. $13,997$ (two sig figs)

22. $13,997$ (three sig figs)

23. 2.596 (two sig figs)

24. 2.596 (three sig figs)

The height of Mount Everest has changed over the years due to plate tectonics and earthquakes. In December 2020, it was jointly announced by Nepal and China that the summit of Mount Everest has an elevation of $29,031.69$ ft.[3]

25. Round $29,031.69$ ft to two sig figs.

26. Round $29,031.69$ ft to three sig figs.

27. Round $29,031.69$ ft to four sig figs.

28. Round $29,031.69$ ft to five sig figs.

29. Round $29,031.69$ ft to six sig figs.

Accuracy when Multiplying and Dividing

Suppose you needed to square the number $3\frac{1}{3}$. You could rewrite $3\frac{1}{3}$ as the improper fraction $\frac{10}{3}$ and then figure out that $(\frac{10}{3})^2 = \frac{100}{9}$, which equals the repeating decimal $11.111...$

Because most people prefer decimals to fractions, we might decide to round $3\frac{1}{3}$ to 3.33 and find that $3.33^2 = 11.0889$. The answer 11.0889 looks very accurate, but it is a false accuracy because there is round-off error involved. Only when we round to three sig figs do we get an accurate result: 11.0889 rounded to three sig figs is 11.1, which is accurate because $11.111...$ rounded to three sig figs is also 11.1. It turns out that because 3.33 has only three significant figures, our answer must be rounded to three significant figures.

When ***multiplying or dividing*** approximate numbers, the answer must be rounded to the same number of significant figures as the ***least*** accurate of the original numbers.

Don't round off the original numbers; do the necessary calculations first, then round the answer as your last step.

Exercises

Use a calculator to multiply or divide as indicated. Then round to the appropriate level of accuracy.

30. $8.75 \cdot 12.25$

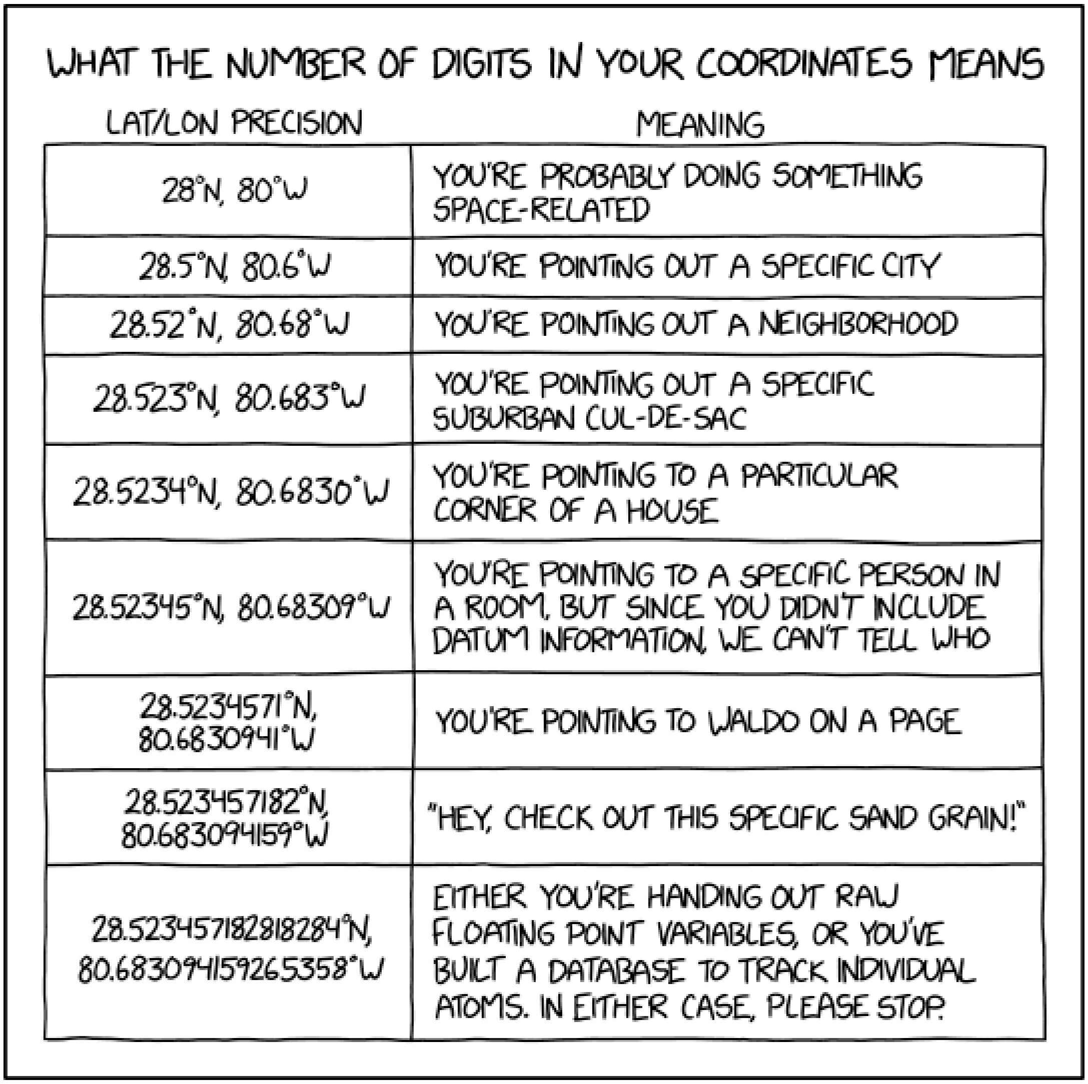
31. $355.12 \cdot 1.8$

32. $77.3 \div 5.375$

33. $53.2 \div 4.5$

34. Suppose you are filling a 5-gallon can of gasoline. The gasoline costs $\$4.579$ per gallon, and you estimate that you will buy 5.0 gallons. How much should you expect to spend?

Bonus material: Here is a comic strip from xkcd.com showing that including a lot of decimal digits can give a false sense of accuracy.



From the web comic xkcd.

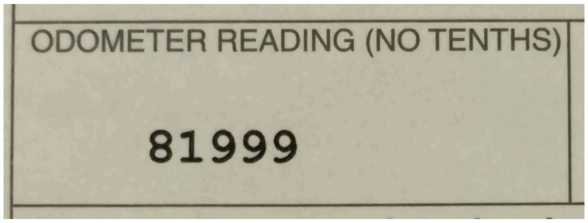
Exercise Answers

Notes

1. Here is an example of a speed camera with gigantic measurement error: https://youtube.com/shorts/cCYy29RuhV8
2. Source: https://en.wikipedia.org/wiki/Mount_Everest#cite_ref-tas1982_33-0
3. Source: https://www.washingtonpost.com/world/asia_pacific/mount-everest-height-nepal-china/2020/12/08/a7b3ad1e-389a-11eb-aad9-8959227280c4_story.html

[6]

Precision and GPE



When someone is selling a used car, its mileage might be listed as 80,000 miles or 82,000 miles because a buyer will want to know the approximate mileage but doesn't need an exact value. If you buy the car, however, you'll need to know the mileage to the nearest mile when you're completing the registration paperwork.

Precision

The ***precision*** of a number is the place value of the rightmost significant figure. For example, $82,000$ is precise to the thousands place, $81,999$ is precise to the ones place, and something like $81,999.2$ would be precise to the tenths place.

Exercises

Identify the precision (i.e., the place value of the rightmost significant figure) of each number.

1. $29,000$

2. $29,\overline{0}00$

3. $29,030$

4. 0.037
5. 0.0307
6. 0.03070

Precision-Based Rounding

In Module 3, we used ***precision-based rounding*** because we were rounding to a specified place value; for example, rounding to the nearest tenth. Let's practice this with overbars and trailing zeros.

Precision-based rounding:

1. Locate the **rounding digit** in the place to which you are rounding.
2. Look at the **test digit** directly to the right of the rounding digit.
3. If the test digit is 5 or greater, increase the rounding digit by 1 and drop all digits to its right. If the test digit is less than 5, keep the rounding digit the same and drop all digits to its right.

Remember, when the rounding digit of a whole number is a 9 that gets rounded up to a 0, we must write an overbar above that 0.

Also, when the rounding digit of a decimal number is a 9 that gets rounded up to a 0, we must include the 0 in that decimal place.

Exercises

Round each number to the indicated place value. Be sure to include an overbar or trailing zeros if necessary.

7. $81,999$ (thousands)

8. $81,999$ (hundreds)

9. $81,999$ (tens)

10. 0.5996 (tenths)

11. 0.5996 (hundredths)

12. 0.5996 (thousandths)

Precision when Adding and Subtracting

Suppose the attendance at a large event is estimated at $25,000$ people, but then you see 3 people leave. Is the new estimate $24,997$? No, because the original estimate was precise only to the nearest thousand. We can't start with an imprecise number and finish with a more precise number. If we estimated that $1,000$ people had left, then we could revise our attendance estimate to $24,000$ because this estimate maintains the same level of precision as our original estimate.

When ***adding or subtracting*** numbers with different levels of precision, the answer must be rounded to the same precision as the ***least*** precise of the original numbers.

Don't round off the original numbers; do the necessary calculations first, then round the answer as your last step.

Exercises

Add or subtract as indicated. Round to the appropriate level of precision.

13. Find the combined weight of four packages with the following weights: 9.7 lb, 13.0 lb, 10.5 lb, 6.1 lb.

14. Find the combined weight of four packages with the following weights: 9.7 lb, 13 lb, 10.5 lb, 6.1 lb.

15. While purchasing renter's insurance, Chandra estimates the value of her insurable possessions at $\$10,200$. After selling some items valued at $\$375$, what would be the revised estimate?

16. Chandra knows that she has roughly $\$840$ in her checking account. After using her debit card to make two purchases of $\$25.95$ and $\$16.38$, how much would she have left in her account?

Before we move on, let's circle back to multiplication and division again. If you are multiplying by an exact number, you can consider this a repeated addition. For example, suppose you measure the weight of an object to be 4.37 ounces and you want to know the weight of three of these objects; multiplying 4.37 times 3 is the same as adding $4.37 + 4.37 + 4.37 = 13.11$ ounces. The answer is still precise to the hundredths place. When working with an exact number, we can assume that it has infinitely many significant figures. (Treat exact numbers like royalty; their accuracy is perfect and it would be an insult to even question it.)

Greatest Possible Measurement Error (GPE)

Suppose you are weighing a dog with a scale that displays the weight rounded to the nearest pound. If the scale says Sir Barks-A-Lot weighs 23 pounds, he could weigh anywhere from 22.5 pounds to almost 23.5 pounds. The true weight could be as much as 0.5 pounds above or below the measured weight, which we could write as 23 ± 0.5.

Now suppose you are weighing Sir Barks-A-Lot with a scale that displays the weight rounded to the nearest tenth of a pound. If the scale says Sir Barks-A-Lot weighs 23.0 pounds, we now know that he could weigh anywhere from 22.95 pounds to almost 23.05 pounds. The true weight could be as much as 0.05 pounds above or below the measured weight, which we could write as 23.0 ± 0.05.

As we increase the level of precision in our measurement, we decrease the ***greatest possible measurement error*** or ***GPE***. The GPE is always one half the precision; if the precision is to the nearest tenth, 0.1, the GPE is half of one tenth, or five hundredths, 0.05. The GPE will always be a 5 in the place to the right of the place value of the number's precision.

Another way to think about the GPE is that it gives the range of values that would round off to the number in question. 23 ± 0.5 tells us a lower value and an upper value. $23 - 0.5 = 22.5$ is the lowest weight that would round up to 23, and $23 + 0.5 = 23.5$ is the upper limit of the weights that would round down to 23. (Yes, 23.5 technically would round up to 24, but it is easier to just say 23.5 instead of 23.49.) Using inequalities, we could represent 23 ± 0.5 as the range of values $22.5 \leq \text{weight} < 23.5$.

Exercises

The attendance at a Portland Thorns match is estimated to be $14,000$ people.

17. What is the precision of this estimate?

18. What is the greatest possible error in this estimate?

A roll of plastic sheeting is 0.00031 inches thick.

19. What is the precision of this measurement?

20. What is the greatest possible error in this measurement?

Plastic sheeting 0.00031 inches thick is referred to as 0.31 mil.

21. What is the precision of this measurement, in mils?

22. What is the greatest possible measurement error, in mils?

Summary of Accuracy, Precision, GPE

Here is a summary of the important terms from Modules 5 & 6. It is easy to get them mixed up, but remembering that "precision" and "place value" both start with "p" can be helpful.

Summary of Terms

Significant figures: the digits in a number that we trust to be correct

Accuracy: the number of significant figures

When multiplying or dividing, we use the accuracy to round the result.

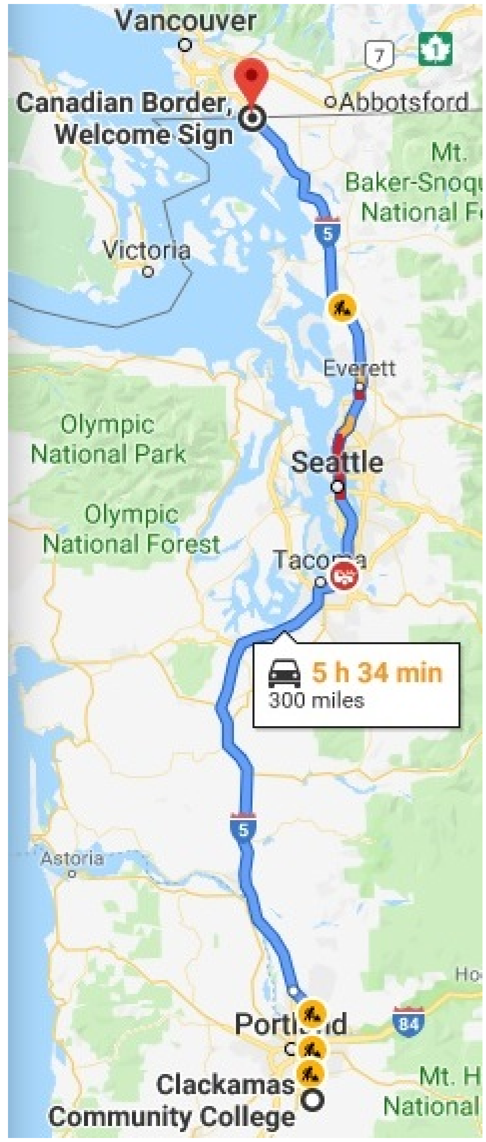
Precision: the place value of the rightmost significant digit

Greatest possible measurement error (GPE): one half the precision

When adding or subtracting, we use the precision to round the result.

Exercises

Google Maps says that the driving distance from CCC's main campus to the Canadian border, rounded to the nearest mile, is 300 miles.



23. Write the distance with an overbar showing the correct precision.

24. What is the precision?

25. What is the GPE?

Google Maps says that the driving time, rounded to the nearest minute, is 5 hours and 34 minutes.

26. What is the precision of this estimate?

27. What is the GPE of this estimate?

Of course, the traffic conditions will change during the trip, making the estimate of 5 hours and 34 minutes unrealistically precise. Let's assume that the drive will take 5.5 hours.

28. What is the *accuracy* of this estimate?

29. What is the *accuracy* of the distance?

30. Calculate the average speed of the vehicle, rounded appropriately.

31. Why did we need to consider the accuracy instead of the precision for this calculation?

Exercise Answers

[7]

Formulas

You may use a calculator throughout this module if needed.

Photo by Rainier Ridao on Unsplash.

A formula is an equation or set of calculations that takes a number (or numbers) as input, and produces an output. The output is often a number, but it could also be a decision such as yes or no.

Each unknown number in a formula is called a ***variable*** because its value can vary. A variable is usually represented with a letter of the alphabet. To evaluate a formula, we substitute a number (or numbers) into the formula and then perform the steps using the order of operations.

Formulas with One Input

Many formulas will have just one input variable. Note: When a number is written directly next to a variable, it indicates multiplication. For example, $0.24w$ means $0.24 \cdot w$.

Exercises

The formula $C = 0.24w + 1.26$ gives the cost, in dollars, of mailing a large envelope weighing w ounces through the USPS.[1]

1. Find the cost of mailing a 6-ounce envelope.
2. Find the cost of mailing a 12-ounce envelope.

Radio Cab charges the following rates for a taxi ride: a fixed fee of $\$3.80$ to get in the taxi, plus a rate of $\$2.80$ per mile.[2] The total cost, in dollars, of a ride m miles long can be represented by the formula $C = 3.80 + 2.80m$.

3. Find the cost of a 5-mile ride.
4. Find the cost of a 7.5-mile ride.
5. Find the cost of getting in the taxi, then changing your mind and getting out without riding anywhere.

The number of members a state has in the U.S. House of Representatives can be approximated by the formula $R = P \div 0.76$, where P is the population in millions.[3] The 2020 populations of three states are as follows:[4]

Oregon	4.2 million
Washington	7.7 million
California	39.6 million

Round all answers to the nearest whole number.

6. How many U.S. Representatives does Oregon have?
7. How many U.S. Representatives does Washington have?
8. How many U.S. Representatives does California have?

The number of electoral votes a state has can be approximated by the formula $E = P \div 0.76 + 2$, where P is the population in millions.

9. How many electoral votes does Oregon have?
10. How many electoral votes does Washington have?
11. How many electoral votes does California have?

Temperature Conversions

You probably know that 32° Fahrenheit is the freezing point of water, and 212° Fahrenheit is the boiling point of water. The Celsius equivalents are 0°C and 100 $^\circ$C. The formulas shown below allow us to convert between the two temperature scales.

Temperature Formulas

$F = \frac{9}{5}C + 32$

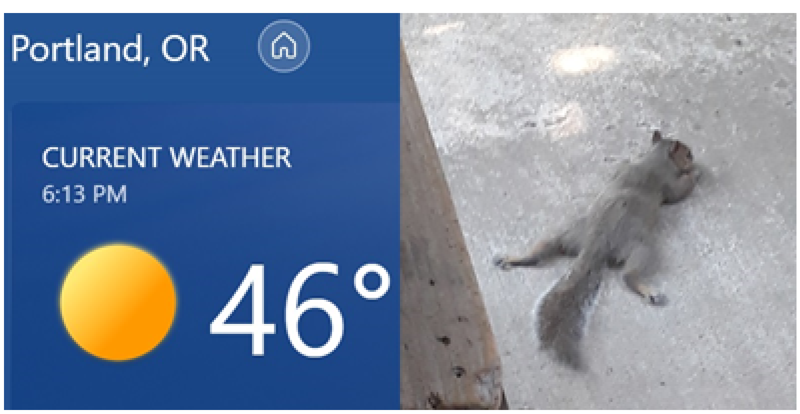
$F = 1.8C + 32$

$C = \frac{5}{9}(F - 32)$

$C = (F - 32) \div 1.8$

Exercises

12. The record high temperature in Portland, Oregon occurred during the "heat dome" event in June, 2021. As shown, it was 46°C at your author's house and the squirrels were laying low. Convert this temperature to Fahrenheit.

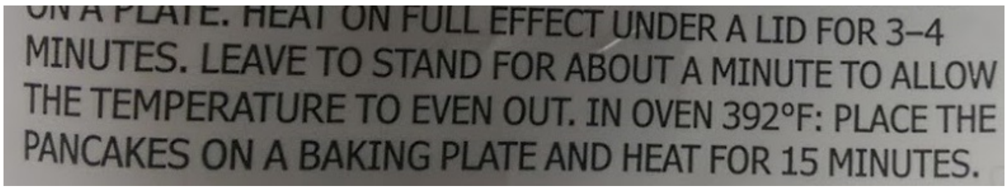


Based on the evidence, the melting point of squirrel is right around 46°C.

13. The traditionally accepted "normal" body temperature of a human[5] is 98.6°F. What is this temperature in Celsius?

14. The FDA recommends that a freezer be set below -18°C. What is the Fahrenheit equivalent?

15. A package of frozen pancakes from IKEA calls for the oven to be set to 392 °F. IKEA is based in Sweden, and this temperature clearly was originally calculated in Celsius. What is the corresponding Celsius temperature?



Formulas with More than One Input

Some formulas have more than one input variable. Just pay attention to which number goes in for each variable.

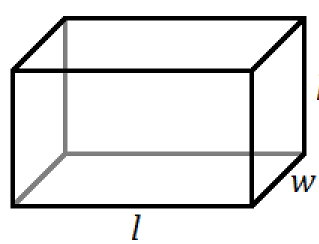
Exercises

When a patient's blood pressure is checked, they are usually told two numbers: the systolic blood pressure (SBP) and the diastolic blood pressure (DBP). The mean arterial pressure (MAP) can be estimated by the following formula: $MAP = \frac{SBP+2 \cdot DBP}{3}$. (The units are mm Hg.) Calculate the mean arterial pressure for each patient.

16. SBP = 120, DBP = 75

17. SBP = 140, DBP = 90

UPS uses the following formula[6] to determine the "measurement" of a package with length l, width w, and height h: $m = l + 2w + 2h$. Determine the measurement of a package with the following dimensions.



18. length 18 inches, width 12 inches, height 14 inches

19. length 16 inches, width 14 inches, height 15 inches

Formulas with a Yes or No Answer

Some formulas give a yes or no answer: success or failure, approved or disapproved, etc. After calculating the result from the formula, we need to compare it to a given number to see whether the result is within a specified range.

Exercises

In Australia, a chicken egg is designated "large" if its mass, in grams, satisfies the following formula: $|m - 54.1| \leq 4.1$. Determine whether each egg qualifies as large.[7]

20. Egg 1's mass is 57.8 grams.

21. Egg 2's mass is 58.3 grams.

22. Egg 3's mass is 49.8 grams.

23. Egg 4's mass is 50.0 grams.

Exercise Answers

Notes

1. Source: https://pe.usps.com/text/dmm300/Notice123.htm#_c037
2. Source: https://www.radiocab.net/services-radio-cab/
3. The value 0.76 comes from dividing the total U.S. population in 2020, around 331 million people, by the 435 seats in the House of Representatives.
4. Source: https://www.census.gov/data/tables/2020/dec/2020-apportionment-data.html
5. See this article for a counterargument: https://www.nytimes.com/2023/10/12/well/live/fever-normal-body-temperature.html
6. Source: https://www.ups.com/us/en/help-center/packaging-and-supplies/prepare-overize.page
7. Source: https://en.wikipedia.org/wiki/Chicken_egg_sizes

[8]

Perimeter and Circumference

You may use a calculator throughout this module if needed.

Perimeter

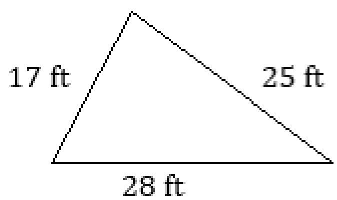
A ***polygon*** is a closed geometric figure with straight sides. Common polygons include triangles, squares, rectangles, parallelograms, trapezoids, pentagons, hexagons, octagons… Just as a perimeter fence runs along the outside edge of a region, the ***perimeter*** of a polygon is the total distance around the outside. In general, to find the perimeter of a polygon, you can add up the lengths of all of its sides.

Photo by Hermes Rivera on Unsplash

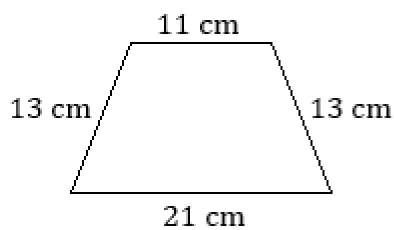
Also, if you haven't already, now is the time to get in the habit of including units in your answers.

Exercises

1. Find the perimeter of the triangle.



2. Find the perimeter of the trapezoid.



If we know that some of the sides of a polygon are equal, we can use a formula as an alternative to adding up all of the lengths individually. The first formula shown below uses the variable s for the side of a square. The rectangle formulas use l for length and w for width, or b for base and h for height; these terms are interchangeable.

Perimeter Formulas

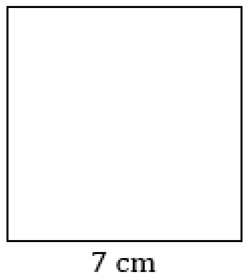
Square: $P = 4s$

Rectangle: $P = 2l + 2w$ or $P = 2b + 2h$

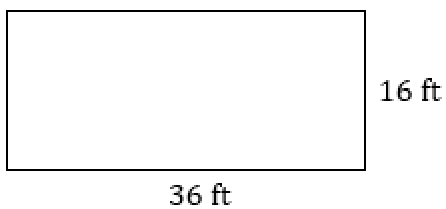
Rectangle: $P = 2(l + w)$ or $P = 2(b + h)$

Exercises

3. Find the perimeter of the square.



4. Find the perimeter of the rectangle.



5. A storage area, which is a rectangle that is 45 feet long and 20 feet wide, needs to be fenced around all four sides. How many feet of fencing is required? (To keep it simple, ignore any gates or other complications.)

6. Giancarlo is putting crown molding around the edge of the ceiling of his living room. If the room is a 12-foot by 16-foot rectangle, how much crown molding does he need?

The sides of a ***regular polygon*** are all equal in length. Therefore, multiplying the length of a side by the number of sides will give us the perimeter.

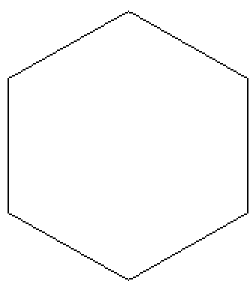
Perimeter Formula

Regular Polygon with n sides of length s: $P = n \cdot s$

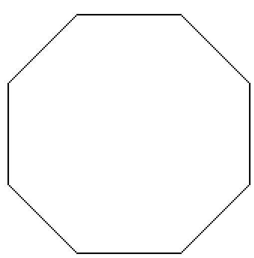
Exercises

Find the perimeter of each regular polygon.

7. Each side of the hexagon is 4 inches long.



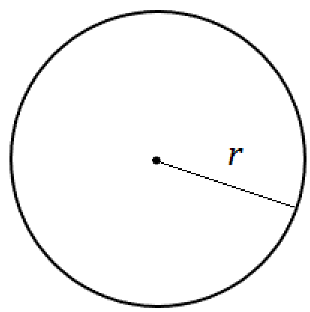
8. Each side of the octagon is 2.5 centimeters long.

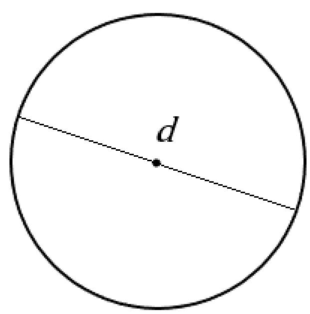


Circumference

The distance around the outside of a circle is called the ***circumference***, rather than the perimeter. Let's review some circle vocabulary before moving on.

Every point on a circle is the same distance from its center. This distance from the center to the edge of the circle is called the ***radius***. The distance from one edge to another, through the center of the circle, is called the ***diameter***. As you can see, the diameter is twice the length of the radius.





Throughout history, different civilizations have discovered that the circumference of a circle is slightly more than 3 times the length of its diameter. (By the year 2000 BCE, the Babylonians were using the value $3\frac{1}{8} = 3.125$ and the Egyptians were using the value $3\frac{13}{81} \approx 3.1605$.)[1] The value $3\frac{1}{7} \approx 3.1429$ is an even better approximation for the ratio of the circumference to the diameter. However, the actual value cannot be written as an exact fraction; it is the irrational number π, pronounced "pie", which is approximately 3.1416.

Circumference Formulas

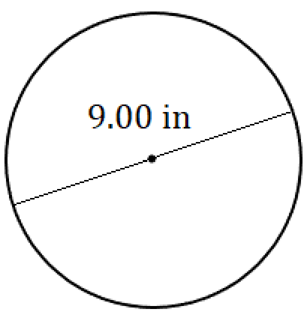
$C = \pi d$

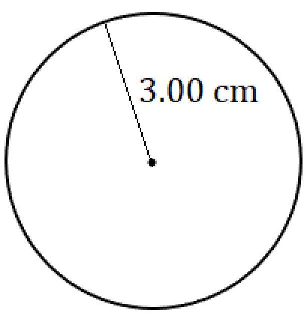
$C = 2\pi r$

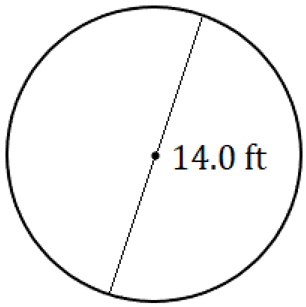
Any scientific calculator will have a π key; using this will give you the most accurate result, although you should be sure to round your answer appropriately. (Remember from Module 5 that we need to pay attention to significant figures when multiplying or dividing.) Many people use 3.14 as an approximation for π, but this can lead to round-off error. If you must use an approximation for π, use 3.1416.

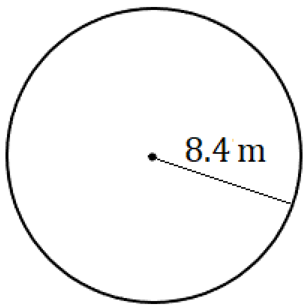
Exercises

Calculate the circumference of each circle. Round each answer to the appropriate level of accuracy.

9. 

10. 

11. 

12. 

Exercise Answers

Notes

1. This information comes from Chapter 1 of the book *A History of Pi* by Petr Beckmann. It is a surprisingly interesting read.

[9]

Percents Part 1

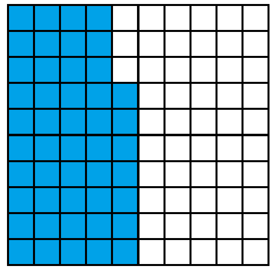
In this module, we will look at the basics of percents, and how percents are related to fractions and decimals. Then we will solve some straightforward percent problems.

We'll return to percents in modules 12 and 27 and solve more complicated problems. This will give you a chance to develop your skills gradually without getting confused.

Percent Basics

The word ***percent*** means "per one hundred". You can think of a percent as a fraction with a denominator of 100.

Exercises



1. What percent of the squares are shaded blue?
2. What percent of the squares are not shaded blue?

To write a percent as a fraction: drop the percent sign, write the number over 100, and simplify if possible. (Notice that if a percent is greater than 100%, the fraction will be greater than 1, and if a percent is less than 1%, the fraction will be less than $\frac{1}{100}$.)

Exercises

Write each percent as a fraction, and simplify if possible.

Photo by NASA on Unsplash

3. About 71% of Earth's surface is covered by water.[1]
4. About 1.3% of Earth's land surface is permanent cropland.[2]
5. About 0.04% of Earth's atmosphere is carbon dioxide.[3]
6. The number of HVACR technician jobs in the U.S. in 2032 is predicted to be 106% of the number of jobs in 2022.[4]

To write a percent as a decimal: drop the percent sign and move the decimal point two places to the left. (Notice that if the percent is not a whole number, the decimal will extend beyond the hundredths place.)

Exercises

Write each percent from Exercises 3 through 6 as a decimal.

7. 71%
8. 1.3%
9. 0.04%
10. 106%

To write a decimal as a percent: move the decimal point two places to the right and insert a percent sign.

Exercises

Write each decimal number as a percent.

11. 0.23
12. 0.07
13. 0.085
14. 2.5

To write a fraction as a percent, write the fraction as a decimal by dividing the numerator by the denominator, then move the decimal point two places to the right and insert a percent sign.

Alternate method: If the denominator of the fraction is a factor of 100, it can easily be built up to have a denominator of 100.

Exercises

15. 7 out of 25 students were tardy on Wednesday. Write $\frac{7}{25}$ as a percent.

16. A package of 24 m&m's contained 3 orange m&m's. Write $\frac{3}{24}$ as a percent.

Solving Percent Problems: Finding the Amount

You may use a calculator for the remainder of this module if needed.

We often use the words ***amount*** and ***base*** in a percent problem. The ***amount*** is the answer we get after finding the percent of the original number. The ***base*** is the original number, the number we find the percent of. (You may also think of the amount as the part, and the base as the whole.) We can call the percent the ***rate***.

$$\text{Amount} = \text{Rate} \cdot \text{Base}$$

$$A = R \cdot B$$

Be sure to change the percent to a decimal before multiplying.

Exercises

17. What is 9% of 350?
18. 30% of 75 is what number?
19. Find 13.5% of 500.
20. 125% of 80 is equal to what amount?
21. What number is 40% of 96.5?
22. Calculate 0.5% of 450.

Suppose you buy an electric drill with a retail price of $\$109.97$ in a city with 8.5% sales tax.

23. Find the amount of the tax. Round to the nearest cent, if necessary.
24. How much do you pay in total?

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/Earth#Surface
2. Source: https://en.wikipedia.org/wiki/Earth#Surface
3. Source: https://en.wikipedia.org/wiki/Atmosphere_of_Earth#Composition
4. Source: Bureau of Labor Statistics, U.S. Department of Labor, Occupational Outlook Handbook, Heating, Air Conditioning, and Refrigeration Mechanics and Installers, at https://www.bls.gov/ooh/installation-maintenance-and-repair/heating-air-conditioning-and-refrigeration-mechanics-and-installers.htm

[10]

Ratios, Rates, Proportions

As we saw in the previous module, a percent is a fraction that compares a number to 100. This is an example of a ratio. In this module, we will focus on ratios and rates, which look like fractions, and then we'll finish up with proportions, which look like two fractions set equal to each other. Many of the examples in this module can be worked without a calculator, while others are best done with a calculator because you'll be multiplying big numbers or dividing messy decimals. You can ask your instructor for guidance if you aren't sure whether a calculator is appropriate.

Ratios & Rates

A ***ratio*** is the quotient of two numbers or the quotient of two quantities with the same units. (Pop quiz… Which operation gives you a quotient: addition, subtraction, multiplication, or division?)

When writing a ratio as a fraction, the first quantity is the numerator and the second quantity is the denominator. If the fraction can be simplified, reduce it to lowest terms.

Exercises

1. Find the ratio of 45 minutes to 2 hours. Simplify the fraction, if possible.

A ***rate*** is the quotient of two quantities with different units. You must include the units.

When writing a rate as a fraction, the first quantity is the numerator and the second quantity is the denominator. If the fraction can be simplified, reduce it to lowest terms.

Exercises

2. A car travels 105 miles in 2 hours. Write the rate as a fraction.

Unit Rates

For practical purposes, expressing a rate as a reduced fraction is not always useful. Expressing a rate as a single number with units such as miles per hour is often more meaningful. This is called a ***unit rate*** because it expresses the quantity in the numerator that corresponds to *one* unit of the denominator.

To find the unit rate, divide the numerator by the denominator and express the rate as a mixed number or decimal. The units can be expressed with the word "per": miles per hour, dollars per gallon, grams per deciliter, pounds per square inch, and so on.

Exercises

3. A car travels 105 miles in 2 hours. Write the car's average speed as a unit rate.

A ***unit price*** is a rate with the price in the numerator and a denominator equal to 1. The unit price tells the cost of one unit or one item. To find the unit price, divide the cost by the size or number of items.

You may use a calculator to calculate the unit price.

Exercises

4. A 15.8-ounce box of Carmella Creeper cereal costs \$5.29. Determine the unit price.

5. An 18.8-ounce box of Count Chocula cereal costs \$5.29. Determine the unit price.

6. The Monster Cereals are on sale, 2 boxes for \$9. If you buy a 16-ounce box of Franken Berry cereal and a 16-ounce box of Boo Berry cereal, what is the unit price per ounce?

Proportions

A ***proportion*** says that two ratios (or rates) are equal.

Exercises

Determine whether each proportion is true or false by simplifying each fraction.

7. $\frac{6}{8} = \frac{21}{28}$

8. $\frac{10}{15} = \frac{16}{20}$

A common method of determining whether a proportion is true or false is called ***cross-multiplying*** or finding the ***cross products***. We multiply diagonally across the equal sign. In a true proportion, the cross products are equal.

$$\frac{a}{b} = \frac{c}{d} \rightarrow \ a \cdot d = b \cdot c$$

Exercises

Determine whether each proportion is true or false by cross-multiplying.

9. $\frac{6}{8} = \frac{21}{28}$

10. $\frac{10}{15} = \frac{16}{20}$

11. $\frac{14}{4} = \frac{15}{5}$

12. $\frac{0.8}{4} = \frac{5}{25}$

As we saw in Module 7, we can use a ***variable*** to stand for a missing number. If a proportion has a missing number, we can use cross multiplication to solve for the missing number. This is as close to algebra as we get in this textbook.

To solve a proportion for a variable:

1. Set the cross products equal to form an equation of the form $a \cdot d = b \cdot c$.
2. Isolate the variable by rewriting the multiplication equation as a division equation.
3. Check the solution by substituting the answer into the original proportion and finding the cross products.

You may discover slightly different methods that you prefer. If you think "Hey, can't I do this a different way?", you're probably right.

Exercises

Solve for the variable.

13. $\frac{8}{10} = \frac{x}{15}$

14. $\frac{3}{2} = \frac{7.5}{n}$

15. $\frac{3}{k} = \frac{18}{24}$

16. $\frac{w}{6} = \frac{15}{9}$

17. $\frac{5}{4} = \frac{13}{x}$

18. $\frac{3.2}{7.2} = \frac{m}{4.5}$ (calculator recommended)

Problems that involve rates, ratios, scale models, etc. can be solved with proportions. When solving a real-world problem using a proportion, be consistent with the units.

You may use a calculator for the remainder of this module if needed.

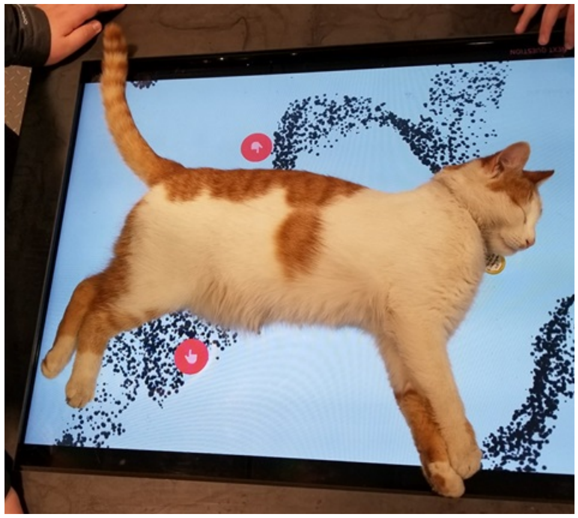
Exercises

19. Tonisha drove her car 320 miles and used 12.5 gallons of gas. At this rate, how far could she drive using 10 gallons of gas?

20. Marcus worked 14 hours and earned $\$210$. At the same rate of pay, how long would he have to work to earn $\$300$?

21. Taylor is trying to figure out the straight-line distance from Portland to Mt. Hood. On their map, $\frac{3}{4}$ inch represents 5 miles. The distance between Portland and Mt. Hood on the map is pretty close to 7 inches. What is the actual distance?

22. Púki the cat lives in a Reykjavík bookstore and likes to sleep on a warm tabletop video screen. Suppose you have a picture of Púki that is 285 pixels wide and 255 pixels high but you need to reduce it in size so that it is 170 pixels high. If the height and width are kept proportional, what is the width of the picture after it has been reduced?



That’s pronounced “pookee”, not “pyookee”.

Exercise Answers

[11]

Scientific Notation

"Number 10" by yoppy is licensed under CC BY 2.0

Powers of Ten

Decimal notation is based on powers of 10: 0.1 is $\frac{1}{10^1}$, 0.01 is $\frac{1}{10^2}$, 0.001 is $\frac{1}{10^3}$, and so on.

We represent these powers with negative exponents: $\frac{1}{10^1} = 10^{-1}$, $\frac{1}{10^2} = 10^{-2}$, $\frac{1}{10^3} = 10^{-3}$, etc.

Negative exponents: $\frac{1}{10^n} = 10^{-n}$

Note: This is true for any base, not just 10, but we will focus only on 10 in this course.

With our base 10 number system, any power of 10 can be written as a 1 in a certain decimal place.

10^4	10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$10,000$	$1,000$	100	10	1	0.1	0.01	0.001	0.0001

If you haven't watched the video "Powers of Ten" from 1977 on YouTube, take ten minutes right now and check it out. Your mind will never be the same again.

Scientific Notation

Let's consider how we could rewrite some different numbers using these powers of 10.

Let's take 50,000 as an example. 50,000 is equal to $5 \times 10,000$ or 5×10^4.

Looking in the other direction, a decimal such as 0.0007 is equal to 7×0.0001 or 7×10^{-4}.

The idea behind scientific notation is that we can represent very large or very small numbers in a more compact format: a number between 1 and 10, multiplied by a power of 10.

> A number is written in scientific notation if it is written in the form $a \times 10^n$, where n is an integer and a is any real number such that $1 \leq a < 10$.

Note: An **integer** is a number with no fraction or decimal part: … -3, -2, -1, 0, 1, 2, 3 …

Although we generally try to avoid using the "x" shaped multiplication symbol, it is frequently used with scientific notation.

Exercises

1. Earth and Mars are two of the smaller planets. Earth has a mass of approximately 5,970,000,000,000,000,000,000,000 kilograms, and Mars has a mass of approximately 639,000,000,000,000,000,000,000 kilograms. Can you determine which mass is larger?

Clearly, it is difficult to keep track of all those zeros. Let's rewrite those huge numbers using scientific notation.

Exercises

2. Earth has a mass of approximately 5.97×10^{24} kilograms, and Mars has a mass of approximately 6.39×10^{23} kilograms. Can you determine which mass is larger?

It is much easier to compare the powers of 10 and determine that the mass of the Earth is larger because it has a larger power of 10. You may be familiar with the term ***order of magnitude***; this simply refers to the difference in the powers of 10 of the two numbers. Earth's mass is one order of magnitude larger because 24 is 1 more than 23.

Exercises

Suppose someone tells you their salary is "six figures".

3. To the nearest dollar, what is their minimum possible salary? Write the answer in standard notation and in scientific notation.

4. To the nearest dollar, what is their maximum possible salary? Write the answer in standard notation and in scientific notation.

We can apply scientific notation to small decimals as well.

Exercises

5. The radius of a hydrogen atom is approximately 0.000000000053 meters. The radius of a chlorine atom is approximately 0.00000000018 meters. Can you determine which radius is larger?

Again, keeping track of all those zeros is a chore. Let's rewrite those decimal numbers using scientific notation.

Exercises

6. The radius of a hydrogen atom is approximately 5.3×10^{-11} meters. The radius of a chlorine atom is approximately 1.8×10^{-10} meters. Can you determine which radius is larger?

The radius of the chlorine atom is larger because it has a larger power of 10; the digits 1 and 8 for chlorine begin in the tenth decimal place, but the digits 5 and 3 for hydrogen begin in the eleventh decimal place.

Scientific notation is very helpful for really large numbers, like the mass of a planet, or really small numbers, like the radius of an atom. It allows us to do calculations or compare numbers without going cross-eyed counting all those zeros.

Exercises

Write each of the following numbers in scientific notation.

7. $1,234$
8. $10,200,000$

9. 0.000870

10. 0.0732

Convert the following numbers from scientific notation to standard decimal notation.

11. 3.5×10^4

12. 9.012×10^7

13. 8.25×10^{-3}

14. 1.4×10^{-5}

Multiplying & Dividing with Scientific Notation

You may be familiar with a shortcut for multiplying numbers with zeros on the end; for example, to multiply $300 \times 4,000$, we can multiply the significant digits $3 \times 4 = 12$ and count up the total number of zeros, which is five, and write five zeros on the back end of the 12: $1,200,000$. This shortcut can be applied to numbers in scientific notation.

To multiply powers of 10, add the exponents: $10^m \cdot 10^n = 10^{m+n}$

Exercises

Multiply each of the following and write the answer in scientific notation.

15. $(2 \times 10^3)(4 \times 10^4)$

16. $(5 \times 10^4)(7 \times 10^8)$

17. $(3 \times 10^{-2})(2 \times 10^{-3})$

18. $(8 \times 10^{-5})(6 \times 10^9)$

When the numbers get messy, it's probably a good idea to use a calculator. If you are dividing numbers in scientific notation with a calculator, you may need to use parentheses carefully. The following rule is true, but you may just want to use a calculator instead.

To divide powers of 10, subtract the exponents: $10^m \div 10^n = 10^{m-n}$

Exercises

19. New Jersey has the highest population density of the 50 states.[1] Its population is 9.29×10^6 people and its land area is 7.35×10^3 square miles. Divide these numbers to find the population density in people per square mile.

20. California has the highest population density of all states west of the Mississippi River.[2] Its population is 3.90×10^7 people and its land area is 1.56×10^5 square miles. Divide these numbers to find the population density in people per square mile.

21. The mass of a proton is 1.67×10^{-27} kg. The mass of an electron is 9.11×10^{-31} kg. Divide these numbers to determine approximately how many times greater the mass of a proton is than the mass of an electron.

Engineering Notation

Closely related to scientific notation is ***engineering notation***, which uses powers of $1,000$. This is the way large numbers are often reported in the news; if roughly $37,000$ people live in Oregon City, we say "thirty-seven thousand" and we might see it written as "37 thousand"; it would be unusual to think of it as $3.7 \times 10,000$ and report the number as "three point seven ten thousands".

One thousand = 10^3, one million = 10^6, one billion = 10^9, one trillion = 10^{12}, and so on.

In engineering notation, the power of 10 is always a multiple of 3, and the other part of the number must be between 1 and $1,000$.

A number is written in engineering notation if it is written in the form $a \times 10^n$, where n is a multiple of 3 and a is any real number such that $1 \leq a < 1,000$.

Note: Prefixes for large numbers such as kilo, mega, giga, and tera are essentially engineering notation, as are prefixes for small numbers such as micro, nano, and pico. We'll see these in Module 16.

Exercises

Write each number in engineering notation, then in scientific notation.

22. The U.S. population is around 335.9 million people.[3]

23. The world population is around 8.020 billion people.[4]

24. The U.S. national debt is around 33.9 trillion dollars.[5]

---

25. Divide the U.S. national debt by the U.S. population to determine the amount of debt per person.

For a visualization of the relative sizes of a million, a billion, and a trillion, see this graphic made by Chris Kane, graphic artist and my friend from high school (PDF file).

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/New_Jersey
2. Source: https://en.wikipedia.org/wiki/List_of_states_and_territories_of_the_United_States_by_population_density
3. Retrieved from *https://www.census.gov/popclock/*, January 1, 2024

4. Retrieved from *https://www.census.gov/popclock/*, January 1, 2024
5. Retrieved from *https://fiscaldata.treasury.gov/datasets/debt-to-the-penny/*, January 1, 2024

[12]

Percents Part 2 and Error Analysis

You may use a calculator throughout this module.

How could anyone pass up that 0.005% discount?

Recall: The ***amount*** is the answer we get after finding the percent of the original number. The ***base*** is the original number, the number we find the percent of. We can call the percent the ***rate***.

When we looked at percents in Module 9, we focused on finding the amount. In this module, we will practice finding the percentage rate and the base.

$$\text{Amount} = \text{Rate} \cdot \text{Base}$$

$$A = R \cdot B$$

We can translate from words into algebra.

- "is" means equals
- "of" means multiply
- "what" means a variable

Solving Percent Problems: Finding the Rate

Suppose you earned 56 points on a 60-point quiz. To figure out your grade as a percent, you need to answer the question "56 is what percent of 60?" We can translate this sentence into the equation $56 = R \cdot 60$.

Exercises

1. 56 is what percent of 60?
2. What percent of 120 is 45?

Be aware that this method gives us the answer in decimal form and we must move the decimal point to convert the answer to a percent.

Also, if the instructions don't explicitly tell you how to round your answer, use your best judgment: to the nearest whole percent or nearest tenth of a percent, to two or three significant figures, etc.

Solving Percent Problems: Finding the Base

Suppose you earn 2% cash rewards for the amount you charge on your credit card. If you want to earn $\$50$ in cash rewards, how much do you need to charge on your card? To figure this out, you need to answer the question "50 is 2% of what number?" We can translate this into the equation $50 = 0.02 \cdot B$.

Exercises

3. $\$50$ is 2% of what dollar value?
4. 5% of what number is 36?

Solving Percent Problems: Using Proportions

Recall that a percent is a ratio, a fraction out of 100. Instead of translating word for word as we just were, we can set up a proportion with the percentage rate over 100. Because the base is the original amount, it corresponds to 100%.

$$\frac{\text{amount}}{\text{base}} = \frac{\text{percent}}{100}$$

Let's try Exercises 1 through 4 again, using proportions.

Exercises

5. 56 is what percent of 60?
6. What percent of 120 is 45?
7. $\$50$ is 2% of what dollar value?
8. 5% of what number is 36?

Now that we have looked at both methods, you are free to use whichever method you prefer: percent equations or proportions.

Exercises

9. The University of Oregon women's basketball team made 13 of the 29 three-points shots they attempted during a game against UNC. What percent of their three-point shots did the team make?
10. A bottle of Dr. Pepper contains 65 grams of added sugars, which is 129% of the recommended daily intake.[1] What is the recommended daily intake?

Solving Percent Problems: Percent Increase

When a quantity changes, it is often useful to know by what percent it changed. If the price of a candy bar is increased by 50 cents, you might be annoyed because it's it's a relatively large percentage of the original price. If the price of a car is increased by 50 cents, though, you wouldn't care because it's such a small percentage of the original price.

To find the percent of increase:

1. Subtract the two numbers to find the amount of increase.
2. Using this result as the amount and the **original** number as the base, divide and find the unknown percent.

Notice that we always use the ***original*** number for the base, the number that occurred earlier in time. In the case of a percent increase, this is the smaller of the two numbers.

Exercises

11. The price of a candy bar increased from $0.89 to $1.39. By what percent did the price increase?

12. Your author bought an overpriced t-shirt at a Seattle Kraken hockey game with a retail price of $40.00. Including sales tax, the actual cost was $44.04. What was the sales tax rate?

Solving Percent Problems: Percent Decrease

Finding the percent decrease in a number is very similar.

To find the percent of decrease:

1. Subtract the two numbers to find the amount of decrease.
2. Using this result as the amount and the **original** number as the base, divide and find the unknown percent.

Again, we always use the ***original*** number for the base, the number that occurred earlier in time. For a percent decrease, this is the larger of the two numbers.

Exercises

13. During a sale, the price of a candy bar was reduced from $\$1.39$ to $\$0.89$. By what percent did the price decrease?

14. The estimated population of Portland, Oregon in April 2020 was $652,500$. The estimated population in July 2022 was $635,100$. Find the percent of decrease in the population, to the nearest tenth of a percent.[2]

To summarize, we can determine the percent change using the following formula, which works whether we're finding a percent increase or a percent decrease.

$$\text{percent change} = \frac{|\text{new} - \text{original}|}{\text{original}} \cdot 100$$

Relative Error

In Module 5, we said that a measurement will always include some error, no matter how carefully we measure. It can be helpful to consider the size of the error relative to the size of what is being measured. As we saw in the examples above, a difference of 50 cents is important when we're pricing candy bars but insignificant when we're pricing cars. In the same way, an error of an eighth of an inch could be a deal-breaker when you're trying to fit a screen into a window frame, but an eighth of an inch is insignificant when you're measuring the length of your garage.

The ***expected outcome*** is what the number would be in a perfect world. If a window screen is supposed to be exactly 25 inches wide, we call this the expected outcome, and we treat it as though it has infinitely many significant digits. In theory, the expected outcome is 25.000000...

To find the ***absolute error***, we subtract the measurement and the expected outcome. Because we always treat the expected outcome as though it has unlimited significant figures, the absolute error should have the same precision (place value) as the *measurement*, not the expected outcome.

To find the ***relative error***, we divide the absolute error by the expected outcome. We usually express the relative error as a percent. In fact, the procedure for finding the relative error is identical to the procedures for finding a percent increase or percent decrease!

To find the relative error:

1. Subtract the two numbers to find the absolute error.
2. Using the **absolute error** as the amount and the **expected outcome** as the base, divide and find the unknown percent.

Exercises

15. A window screen is measured to be $25\frac{3}{16}$ inches wide instead of the advertised 25 inches. Determine the relative error, rounded to the nearest tenth of a percent.

16. The contents of a box of cereal are supposed to weigh 10.8 ounces, but they are measured at 10.67 ounces. Determine the relative error, rounded to the nearest tenth of a percent.

The following formula has the same structure as the percent change formula we saw earlier.

$$\text{percent error} = \frac{|\text{measured} - \text{expected}|}{\text{expected}} \cdot 100$$

Tolerance

The ***tolerance*** is the maximum amount that a measurement is allowed to differ from the expected outcome. For example, the U.S. Mint needs its coins to have a consistent size and weight so that they will work in vending machines. A dime (10 cents) weighs 2.268 grams, with a tolerance of ± 0.091 grams.[3] This tells us that the minimum acceptable weight is $2.268 - 0.091 = 2.177$ grams, and the maximum acceptable weight is $2.268 + 0.091 = 2.359$ grams. A dime with a weight outside of the range $2.177 \le \text{weight} \le 2.359$ would be unacceptable.

Exercises

A U.S. nickel (5 cents) weighs 5.000 grams with a tolerance of ± 0.194 grams.

17. Determine the lowest acceptable weight and highest acceptable weight of a nickel.

18. Determine the relative error of the weight of a nickel with an absolute error of 0.194 grams.

A U.S. quarter (25 cents) weighs 5.670 grams with a tolerance of ± 0.227 grams.

19. Determine the lowest acceptable weight and highest acceptable weight of a quarter.

20. Determine the relative error of the weight of a quarter with an absolute error of 0.227 grams.

Exercise Answers

Notes

1. Source (PDF file): https://www.fda.gov/media/135299/download
2. Source: https://www.census.gov/quickfacts/fact/table/portlandcityoregon/PST045222
3. Sources: https://www.usmint.gov/learn/coin-and-medal-programs/coin-specifications and https://www.thesprucecrafts.com/how-much-do-coins-weigh-4171330

[13]

The US Measurement System

You may use a calculator throughout this module if needed.

Robin the cat has the spirit of '76.

The U.S. customary system of measurement developed from the system used in England centuries ago.[1] To convert from one unit to another, we often have to perform messy calculations like dividing by 16 or multiplying by $5,280$.

We could solve these unit conversions using proportions, but there is another method that is more versatile, especially when a conversion requires more than one step. This method goes by various names, such as *dimensional analysis* or the *factor label method*. The basic idea is to begin with the measurement you know, then multiply it by a conversion ratio that will cancel the units you don't want and replace it with the units you do want.

It's okay if you don't have the conversion ratios memorized; just be sure to have them available. If you discover other conversion ratios that aren't provided here, go ahead and write them down!

U.S. System: Measurements of Length

1 foot = 12 inches

1 yard = 3 feet

1 mile = $5,280$ feet

Let's walk through two examples to demonstrate the process.

Suppose you're a fan of Eminem[2] or the Byrds[3] and you're curious about how many feet are in 8 miles. We can start by writing 8 mi as a fraction over 1 and then use the conversion ratio $1 \text{ mi} = 5,280 \text{ ft}$ to cancel the units.

$$\frac{8 \text{ mi}}{1} \cdot \frac{5280 \text{ ft}}{1 \text{ mi}} = 8 \cdot 5280 \text{ ft} = 42,240 \text{ ft}$$

Now suppose that you want to convert a measurement from feet to miles. (Maybe you're watching *The Twilight Zone* episode "Nightmare at 20,000 Feet"[4] and wondering how many miles high that is.) We'll start by writing 20000 ft as a fraction over 1 and then use the conversion ratio $1 \text{ mi} = 5,280 \text{ ft}$ to cancel the units.

$$\frac{20000 \text{ ft}}{1} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} = \frac{20000}{5280} \text{ mi} \approx 3.8 \text{ mi}$$

As it happens, the first situation became a multiplication problem but the second situation became a division problem. Rather than trying to memorize rules about when you'll multiply versus when you'll divide, just set up the conversion ratio so the units will cancel out and then the locations of the numbers will tell you whether you need to multiply or divide them.

Exercises

1. How many inches are in 4.5 feet?
2. How many feet make up 18 yards?
3. 1 yard is equal to how many inches?

4. 1 mile is equivalent to how many yards?
5. How many feet is 176 inches?
6. 45 feet is what length in yards?
7. Convert $10,560$ feet into miles.
8. How many yards are the same as $1,080$ inches?

Notice that Exercises 3 & 4 give us two more conversion ratios that we could add to our list.

U.S. System: Measurements of Weight or Mass

1 pound = 16 ounces

1 ton = $2,000$ pounds

The procedure is the same; start with the measurement you know and write it as a fraction over 1. Then write the conversion factor so that the units you don't want will cancel out.

Exercises

9. How many ounces are in 2.5 pounds?
10. How many pounds are equivalent to 1.2 tons?
11. Convert 300 ounces to pounds.
12. 1 ton is equivalent to what number of ounces?

U.S. System: Measurements of Volume or Capacity

1 cup = 8 fluid ounces

1 pint = 2 cups

1 quart = 2 pints

1 gallon = 4 quarts

There are plenty of other conversions that could be provided, such as the number of fluid ounces in a gallon, but let's keep the list relatively short.

Exercises

13. How many fluid ounces are in 6 cups?

14. How many pints are in 3.5 quarts?

15. 1 gallon is equal to how many pints?

16. How many cups equal 1.25 quarts?

17. Convert 20 cups into gallons.

18. How many fluid ounces are in one half gallon?

U.S. System: Using Mixed Units of Measurement

Measurements are frequently given with mixed units, such as a person's height being given as 5 ft 7 in instead of 67 in, or a newborn baby's weight being given as 8 lb 3 oz instead of 131 oz. This can sometimes make the calculations more complicated, but if you can convert between improper fractions and mixed numbers, you can handle this.

Exercises

19. A bag of apples weighs 55 ounces. What is its weight in pounds and ounces?

20. A carton of orange juice contains 59 fluid ounces. Determine its volume in cups and fluid ounces.

21. A hallway is 182 inches long. Give its length in feet and inches.

22. The maximum loaded weight of a Ford F-150 pickup truck is $8,500$ lb. Convert this weight into tons and pounds.

We'll finish up this module by adding and subtracting with mixed units. Again, it may help to think of them as mixed numbers, with a whole number part and a fractional part.

Exercises

Comet weighs 8 lb 7 oz and Fred weighs 11 lb 9 oz.

23. Comet and Fred are being put into a cat carrier together. What is their combined weight?

24. How much heavier is Fred than Comet?

Two tables are 5 ft 3 in long and 3 ft 10 in long.

25. If the two tables are placed end to end, what is their combined length?

26. What is the difference in length between the two tables?

Exercise Answers

Notes

1. Want some Wikipedia rabbit holes? Visit https://en.wikipedia.org/wiki/United_States_customary_units and https://en.wikipedia.org/wiki/English_units.
2. https://en.wikipedia.org/wiki/8_Mile_(film)
3. https://en.wikipedia.org/wiki/Eight_Miles_High
4. https://en.wikipedia.org/wiki/Nightmare_at_20,000_Feet

[14]

The Metric System

You will NOT need a calculator for this module.

The metric system was first implemented following the French Revolution; if we're overthrowing the monarchy, why should we use a unit of a "foot" that is based on the length of a king's foot?

The metric system was designed to be based on the natural world, and different units are related to each other by powers of 10 instead of weird numbers like 3, 12, 16, and $5,280$... This makes converting between metric units incredibly easy because all we need to do is move the decimal point.

The table below shows the most common metric prefixes. The prefixes are arranged in order so that we can convert between them simply by moving the decimal point the same number of places shown in the table.

kilo- (k)	hecta- (h)	deka- (da)	**[base unit]**	deci- (d)	centi- (c)	milli- (m)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1{,}000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

Because deka- and deci- both start with d, the abbreviation for deka- is da.

Metric System: Measurements of Length

The base unit of length is the ***meter***, which is a bit longer than a yard (three feet). Because the prefix ***kilo-*** means one thousand, 1 kilometer is $1,000$ meters. (One kilometer is around six tenths of a mile.) Similarly, because the prefix ***centi-*** means one hundredth, 1 centimeter is $\frac{1}{100}$ of a meter, or 1 meter is 100 centimeters. (One centimeter is roughly the thickness of a pen.) And because the prefix ***milli-*** means one thousandth, 1 millimeter is $\frac{1}{1,000}$ of a meter, or 1 meter is $1,000$ millimeters. (One millimeter is roughly the thickness of a credit card.)

Exercises

From each of the four choices, choose the most reasonable measure.

1. The length of a car:
   5 kilometers, 5 meters, 5 centimeters, 5 millimeters
2. The height of a notebook:
   28 kilometers, 28 meters, 28 centimeters, 28 millimeters
3. The distance to the next town:
   3.8 kilometers, 3.8 meters, 3.8 centimeters, 3.8 millimeters
4. An adult woman's height:
   1.6 kilometers, 1.6 meters, 1.6 centimeters, 1.6 millimeters
5. An adult woman's height:
   160 kilometers, 160 meters, 160 centimeters, 160 millimeters
6. The thickness of a pane of glass:
   3 kilometers, 3 meters, 3 centimeters, 3 millimeters

kilo- (km)	hecta- (hm)	deka- (dam)	**meter (m)**	deci- (dm)	centi- (cm)	milli- (mm)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

To convert metric units, you can simply move the decimal point left or right the number of places indicated in the table above. No calculator required!

Exercises

A 2024 Chevrolet Silverado 1500 pickup truck is 5.36 meters long.[1]

7. Convert 5.36 meters to centimeters.

8. Convert 5.36 meters to millimeters.

One mile is approximately 1.609 kilometers.

9. Convert 1.609 kilometers to meters.

10. Convert 1.609 kilometers to centimeters.

A sheet of A4 paper[2] is 297 millimeters long.

11. Convert 297 millimeters to meters.

12. Convert 297 millimeters to centimeters.

The Burj Khalifa in Dubai is the world's tallest building, with a height of 828 meters.[3]

13. Convert 828 meters to kilometers.

14. Convert 828 meters to dekameters.

Metric System: Measurements of Weight or Mass

The base unit for mass is the ***gram***, which is about the mass of a paper clip. A ***kilogram*** is $1,000$ grams; as we'll see in the next module, this is around 2.2 pounds. The active ingredients in medicines may be measured using the ***milligram***.

Exercises

From each of the three choices, choose the most reasonable measure.

15. The mass of an apple:
100 kilograms, 100 grams, 100 milligrams

16. The mass of an adult man:
80 kilograms, 80 grams, 80 milligrams

17. The amount of active ingredient in a pain relief pill:
500 kilograms, 500 grams, 500 milligrams

18. The base vehicle weight of a Chevrolet Silverado 1500 pickup truck:
$2,000$ kilograms, $2,000$ grams, $2,000$ milligrams

kilo- (kg)	hecta- (hg)	deka- (dag)	**gram (g)**	deci- (dg)	centi- (cg)	milli- (mg)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1{,}000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

This table is identical to the previous table; the only difference is that the base unit "meter" has been replaced by "gram". This means that converting metric units of mass is exactly the same process as converting metric units of length.

Exercises

A five-pound bag of flour weighs about 2.27 kilograms.

19. Convert 2.27 kilograms to grams.

20. Convert 2.27 kilograms to milligrams.

A 20-ounce bottle of Dr. Pepper contains 65 grams of sugars.

21. Convert 65 grams to centigrams.

22. Convert 65 grams to milligrams.

23. Convert 65 grams to kilograms.

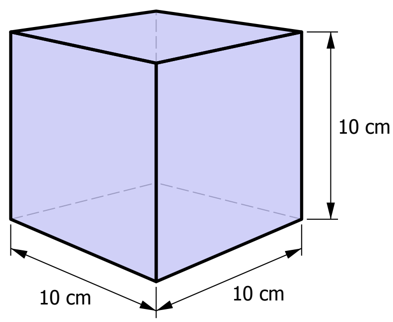
A 20-ounce bottle of Dr. Pepper contains 95 milligrams of sodium.

24. Convert 95 milligrams to centigrams.

25. Convert 95 milligrams to grams.

Metric System: Measurements of Volume or Capacity

The base unit of volume is the ***liter***, which is slightly larger than one quart. The ***milliliter*** is also commonly used; of course, there are $1,000$ milliliters in 1 liter.



1 liter is equivalent to a cube with sides of 10 centimeters. Image adapted by Cristianrodenas on Wikimedia Commons.

In case you were wondering, the units of volume, length, and mass are all connected; one cubic centimeter (a cube with each side equal to 1 cm) has the same volume as one milliliter, and one milliliter of water has a mass of one gram.

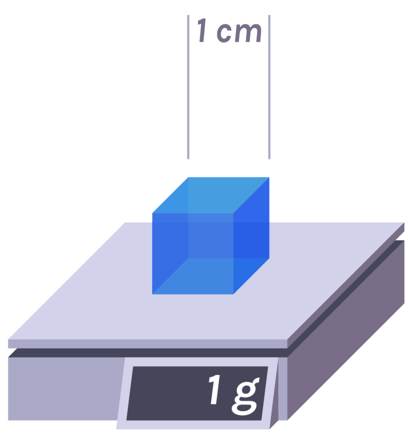


Image by nclm on Wikimedia Commons.

Exercises

From each of the two choices, choose the more reasonable measure.

26. The capacity of a car's gas tank: 50 liters, 50 milliliters

27. A dosage of liquid cough medicine: 30 liters, 30 milliliters

kilo- (kL)	hecta- (hL)	deka- (daL)	**liter (L)**	deci- (dL)	centi- (cL)	milli- (mL)
$1,000$	100	10	1	0.1	0.01	0.001
$1,000$	100	10	1	$\frac{1}{10}$	$\frac{1}{100}$	$\frac{1}{1,000}$
10^3	10^2	10^1	10^0	10^{-1}	10^{-2}	10^{-3}

Again, this table is identical to the previous tables; just move the decimal point left or right to convert the units.

Exercises

28. A bottle of sparkling water is labeled 50 cl. Convert 50 centiliters to liters.

29. A "1-liter" bag of saline solution for intravenous use actually contains about 1.05 liters of solution.[4] How many deciliters is this?

30. A carton of orange juice has a volume of 1.75 liters. Convert this into mL.

31. One cup (8 fluid ounces) is approximately 250 milliliters. Convert 250 milliliters into liters.

32. While being served drinks on IcelandAir, you notice that one mini bottle is labeled 50 mL, but another mini bottle is labeled 5 cL. How do the two bottles compare in size?

33. The engine displacement of a Yamaha Majesty scooter is 125 cc (cubic centimeters), and the engine displacement of a Chevrolet Spark automobile is 1.4 L (liters). What is the approximate ratio of these engine displacements?

34. How many 500-milliliter bottles of Coke are equivalent to one 2-liter bottle?

Exercise Answers

Notes

1. Source: https://www.caranddriver.com/chevrolet/silverado-1500/specs
2. A4 is a bit narrower and a bit longer than standard letter paper that we use in the U.S. and Canada.
3. Source: https://en.wikipedia.org/wiki/Burj_Khalifa
4. Source: https://pubmed.ncbi.nlm.nih.gov/11096388/

[15]

Converting Between Systems

You may use a calculator throughout this module.

Converting between the U.S. system and metric system is important in today's global economy; like it or not, the metric system is infiltrating our lives.

The numbers in these conversion ratios are usually difficult to work with, so we will use a calculator whenever necessary and pay attention to rounding. If you discover other conversion ratios that aren't provided here, write them down!

Converting Measurements of Length

Some of these conversion ratios are exact, because a yard is defined to be exactly 0.9144 meters, which means that a foot is exactly 0.3048 meters and an inch is exactly 2.54 centimeters.[1] The conversions that are not exact are rounded to four significant figures.

1 in = 2.54 cm

1 ft = 30.48 cm

1 ft = 0.3048 m or 1 m ≈ 3.281 ft

1 yd = 0.9144 m or 1 m ≈ 1.094 yd

1 mi ≈ 1.609 km or 1 km ≈ 0.6214 mi

Let's verify that $1.98\ \mathrm{m}$ is actually $6\ \mathrm{ft}\ 6\ \mathrm{in}$. We can start by writing $1.98\ \mathrm{m}$ as a fraction over 1 and then use the conversion ratio $1\ \mathrm{ft} = 0.3048\ \mathrm{m}$ to cancel the units.

$$\frac{1.98\ \mathrm{m}}{1} \cdot \frac{1\ \mathrm{ft}}{0.3048\ \mathrm{m}} = \frac{1.98}{0.3048}\ \mathrm{ft} \approx 6.49606\ \mathrm{ft}$$

Rounding to three sig figs,[2] that's $6.50\ \mathrm{ft}$, which of course is $6\ \mathrm{ft}\ 6\ \mathrm{in}$.

Let's verify that again using the approximate conversion ratio $1\ \mathrm{m} \approx 3.281\ \mathrm{ft}$.

$$\frac{1.98\ \mathrm{m}}{1} \cdot \frac{3.281\ \mathrm{ft}}{1\ \mathrm{m}} = 1.98 \cdot 3.281\ \mathrm{ft} \approx 6.49638\ \mathrm{ft}$$

Okay, everything looks good; both conversions give us a result that rounds to $6.50\ \mathrm{ft}$. As long as we don't try to keep more than four sig figs in our result, we can use either conversion ratio and get the same result.

Exercises

1. In Canada, if a road in a city does not have a speed limit posted, the default speed limit is $50\ \mathrm{km}$ per hour.[3] Convert $50\ \mathrm{km}$ into miles.
2. A *Star Wars* clone trooper is 6.00 feet tall. Convert this height to cm.
3. An Olympic-size swimming pool is 50.000 meters in length.[4] How many feet is this?
4. A 4-inch paintbrush is labeled 101.6 millimeters. Verify the accuracy of this conversion.

5. An electric fan has an advertised diameter of 20 inches, or 50.0 centimeters. Verify the accuracy of this conversion.

6. Is 21 kilometers equivalent to 13 miles? If not, what is the percent error?

Converting Measurements of Weight or Mass

These conversions are approximate. (Technically, one pound is exactly 0.45359237 kilograms[5], but we'll stick to four significant figures.)

1 oz ≈ 28.35 g

1 kg ≈ 2.205 lb or 1 lb ≈ 0.4536 kg

Exercises

As shown in the picture, a shelving system is rated to hold a total weight of $3,250$ pounds, or 650 pounds on each of its five shelves. The metric equivalents printed on the box are $1,474.1$ kilograms and 294.8 kilograms.

7. Convert 650 pounds into kilograms. Does your answer agree with the number printed on the box?

8. Convert $3,250$ pounds into kilograms. Does your answer agree with the number printed on the box?

---

9. A hand weight weighs 5 kilograms. Convert 5 kilograms to pounds.

10. How many grams is a half pound of ground beef?

11. A gravy recipe calls for 4 ounces of flour. Convert 4 ounces into grams.

12. A smoothie recipe has 50 grams of protein. Convert 50 grams to ounces.

13. In around 2010, the National Collector's Mint (not affiliated with the U.S. Mint) ran a TV commercial selling an imitation $\$50$ gold coin modeled after the U.S. "buffalo" nickel. The commercial made the following claims. *This replica coin is coated in* 31 *milligrams of pure gold! And the price of gold keeps going up; gold is worth about* $\$1,000$ *per ounce! But you can order these fake coins for only* $\$19.95$ *apiece!* What was the approximate dollar value of the gold in one of these coins?

14. In 2023, the National Collector's Mint was still selling the imitation $\$50$ gold "buffalo" nickel, but had cut the price to $\$9.95$. This version of the coin was coated in 14 milligrams of pure gold.[6] The price of gold in 2023 was $\$2,000$ per ounce. What was the approximate dollar value of the gold in one of these coins?

Converting Measurements of Volume or Capacity

These conversion ratios are approximations rounded to four significant figures.

$1 \text{ fl oz} \approx 29.57 \text{ mL}$

$1 \text{ L} \approx 33.81 \text{ fl oz}$

$1 \text{ L} \approx 1.057 \text{ qt}$

$1 \text{ gal} \approx 3.785 \text{ L}$

Exercises

15. Stephanie lives in Vermont and buys her home renovation supplies at *Réno-Dépôt* in Quebec. She buys a toilet that uses 4.8 L of water per flush. How many gallons is this?

16. How many milliliters of drink are in a 12-ounce can?

17. TSA airline regulations limit liquids in carry-on luggage to 100 milliliters.

How many fluid ounces is this? (Round your answer to the nearest tenth.)

18. If you visit an oil change shop, you may notice large boxes of motor oil. (If you teach math, you may even take a picture of one so you can use it in your course materials.) Verify that 6 gallons of motor oil is equivalent to 22.7 liters.

Converting Measurements: Extensions

Let's finish up with some rates that require conversions.

Exercises

Maxine is driving across Canada. Her car has a 14.2-gallon gas tank and gets an average of 26 miles per gallon.

19. Approximately how many kilometers—actually, the Canadian spelling is kilometres. Approximately how many kilometres can she travel on a full tank of gas?

20. Of course, Canada measures gas in liters. Actually, litres. Convert Maxine's mileage rate, 26 miles per gallon, to kilometres per litre.

Exercise Answers

Notes

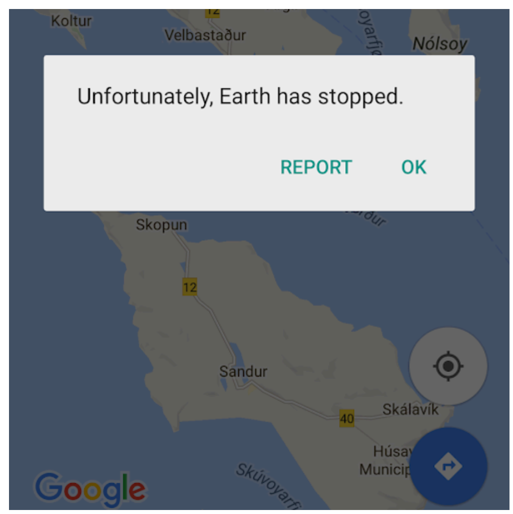
1. Source: https://en.wikipedia.org/wiki/International_yard_and_pound
2. We round to three sig figs because the measurement 1.98 meters has only three sig figs.
3. Source: https://niagarafalls.ca/city-hall/transportation-services/traffic/speed-limits.aspx
4. Source: https://swimswam.com/how-big-is-an-olympic-sized-swimming-pool/
5. Source: https://en.wikipedia.org/wiki/International_yard_and_pound
6. Source: https://ncmint.com/2023-buffalo-tribute-proof/

[16]

Other Conversions

You may use a calculator in this module as needed.

Converting Measurements of Time



You probably know all of the necessary conversions for time: 60 seconds in a minute, 24 hours in a day, etc.

When we get to units of time larger than weeks, however, we encounter problems because not all months have the same number of days, a year is not exactly 52 weeks, and the time it takes for the Earth to orbit the Sun is not exactly 365 days. Therefore, it doesn't make sense to expect an exact answer to a question like "how many minutes are in one month?" We will have to use our best judgment in situations such as these.

1 min = 60 sec

1 hr = 60 min

1 day (dy) = 24 hr

1 week (wk) = 7 dy

1 year (yr) = 365 dy

Exercises

1. How many minutes is one standard 365-day year?
2. Have you been alive for one billion seconds? Is this even possible?

Converting Rates

Previously when we were converting units, we began with units in the numerator only. If we need to convert a rate, however, we'll begin with units in both the numerator and denominator.

Exercises

Usain Bolt holds the world record time for the 100-meter dash, 9.58 seconds.

3. What was his average speed in kilometers per hour?
4. What was his average speed in miles per hour?

The more information we know, the more things we can figure out.

Exercises

An F-15 fighter jet can reach a sustained top speed of roughly Mach 2.3; this is 2.3 times the speed of sound, which is 770 miles per hour.[1]

5. What is the jet's top speed in miles per hour?
6. At this speed, how many miles would the jet travel in one minute?
7. The jet's range at this speed before it runs out of fuel is around 600 miles. If the jet flies 600 miles at top speed, for how many minutes will it fly?
8. The jet's maximum fuel capacity is $3,475$ gallons. If the jet flies 600 miles and burns $3,475$ gallons of fuel, find the jet's fuel efficiency, in miles per gallon.
9. Rewrite the jet's fuel efficiency, in gallons per mile.
10. How many gallons of fuel does the jet consume in one minute?

Measurement Prefixes: Larger

Now let's turn our attention to converting units based on their prefixes. We'll start with large units of measure.

tera- (T)	giga- (G)	mega- (M)	kilo- (k)	**[base unit]**
trillion	billion	million	thousand	one
$1,000,000,000,000$	$1,000,000,000$	$1,000,000$	$1,000$	1
10^{12}	10^9	10^6	10^3	10^0

Notice that the powers of these units are multiples of 3, just as with the engineering notation we saw at the end of Module 11. Each prefix is $1,000$ times the next smaller prefix, so moving one place in the chart is equivalent to moving the decimal point *three* places. Also notice that capitalization is important; megagram (which is also called a metric ton) is Mg with a capital M, but milligram is mg with a lowercase m.

Using computer memory as an example:

1 kilobyte = $1,000$ bytes

1 megabyte = $1,000$ kilobytes = $1,000,000$ bytes

1 gigabyte = $1,000$ megabytes = $1,000,000$ kilobytes, etc.

1 terabyte = $1,000$ gigabytes = $1,000,000$ megabytes, etc.

Note: There can be inconsistencies with different people's understanding of these prefixes with regards to computer memory, which is counted in powers of 2, not 10. Computer engineers originally defined 1 kilobyte as $1,024$ bytes because $2^{10} = 1,024$, which is very close to $1,000$. However, we will consider these prefixes to be powers of $1,000$, not $1,024$. There is an explanation at https://physics.nist.gov/cuu/Units/binary.html.

Exercises

11. A $5\frac{1}{4}$ inch floppy disk from the 1980s could store about 100 kB; a $3\frac{1}{2}$ inch floppy disk from the 1990s could store about 1.44 MB. By what factor was the storage capacity increased?

12. How many times greater is the storage capacity of a 2 TB hard drive than a 500 GB hard drive?

13. In an article describing small nuclear reactors that are designed to be retrofitted into coal plants, Dr. Jose Reyes of Oregon State University says "One module will produce 60 megawatts of electricity. That's enough for about 50 thousand homes."[2] How much electricity per home is this?

14. In the same article, Dr. Reyes says "a 60 megawatt module could produce about 60 million gallons of clean water per day using existing technologies in reverse osmosis." What is the rate of watts per gallon?

15. The destructive power of nuclear weapons is measured in kilotons (the equivalent of $1,000$ tons of TNT) or megatons (the equivalent of $1,000,000$ tons of TNT). The first nuclear device ever tested, the US's *Trinity*, was measured at roughly 20 kilotons on July 16, 1945. The largest thermonuclear weapon ever detonated, at 50 megatons, was the USSR's *Tsar Bomba*, on October 31, 1961.[3] (Video of *Tsar Bomba* was declassified in 2020.) How many times more powerful was *Tsar Bomba* than *Trinity*?

Measurement Prefixes: Smaller

Now we'll go in the other direction and look at small units of measure.

[base unit]	milli- (m)	micro- (μ or mc)	nano- (n)	pico (p)
one	thousandth	millionth	billionth	trillionth
1	0.001	0.000001	0.000000001	0.000000000001
10^0	10^{-3}	10^{-6}	10^{-9}	10^{-12}

The symbol for micro- is the Greek letter μ (pronounced "myoo"). Because this character can be difficult to replicate, you may see the letter "u" standing in for "μ" in web-based or plaintext technical articles… or you may see the prefix "mc" instead.

Again, the powers are multiples of 3; each prefix gets smaller by a factor of $\frac{1}{1000}$.

The negative exponents can sometime be complicated to work with, and it may help to think about things in reverse.

1 meter $= 10^3$ millimeters $= 10^6$ micrometers $= 10^9$ nanometers $= 10^{12}$ picometers

1 second $= 10^3$ milliseconds $= 10^6$ microseconds $= 10^9$ nanoseconds $= 10^{12}$ picoseconds

...and so on.

See https://physics.nist.gov/cuu/Units/prefixes.html for a list of more prefixes.

Exercises

16. An article about network latency compares the following latency times: "So a 10 Mbps link adds 0.4 milliseconds to the RTT, a 100 Mbps link 0.04 ms and a 1 Gbps link just 4 microseconds."[4] Rewrite these times so that they are all in terms of milliseconds, then rewrite them in terms of microseconds.

17. The wavelength of red light is around 700 nm. Infrared radiation has a wavelength of approximately 10 μm.[5] Find the ratio of these wavelengths.

18. Nuclear radiation is measured in units called Sieverts, but because this unit is too large to be practical when discussing people's exposure to radiation, milliSieverts and microSieverts are more commonly used. In 1986, workers cleaning up the Chernobyl disaster were exposed to an estimated dose of 250 mSv.[6] A typical chest x-ray exposes a person to 100 μSv.[7] How many chest x-rays' worth of radiation were the workers exposed to?

Exercise Answers

Notes

1. My sources for the following set of questions are a combination of former students in the Air National Guard and people who sound like they know what they're talking about on the internet, particularly in this Quora discussion.
2. Source: https://www.kgw.com/article/news/local/oregon-company-get-approval-to-build-nuclear-power-plants/283-7b26b8cd-12d5-4116-928a-065731f7a0f6
3. Source: https://en.wikipedia.org/wiki/Nuclear_weapon_yield
4. Source: https://www.noction.com/blog/network-latency-effect-on-application-performance
5. Source: http://labman.phys.utk.edu/phys222core/modules/m6/The%20EM%20spectrum.html
6. Source: https://en.wikipedia.org/wiki/Chernobyl_disaster
7. Source: https://www.cancer.org/treatment/understanding-your-diagnosis/tests/understanding-radiation-risk-from-imaging-tests.html

[17]

Angles

You will need a calculator near the end of this module.

Angle Measure

Angle measurement is important in construction, surveying, physical therapy, and many other fields. We can visualize an angle as the figure formed when two line segments share a common endpoint.

Photo by Rangga Cahya Nugraha on Unsplash

We can also think about an angle as a measure of rotation. One full rotation or a full circle is $360°$, so a half rotation or U-turn is $180°$, and a quarter turn is $90°$. We often classify angles by their size relative to these $90°$ and $180°$ benchmarks.

Acute Angle: between $0°$ and $90°$

Right Angle: exactly $90°$

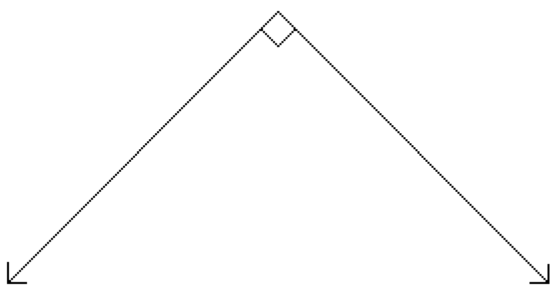
Obtuse Angle: between $90°$ and $180°$

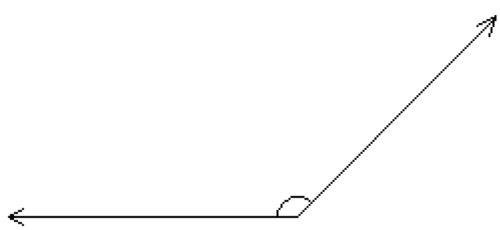
Straight Angle: exactly $180°$

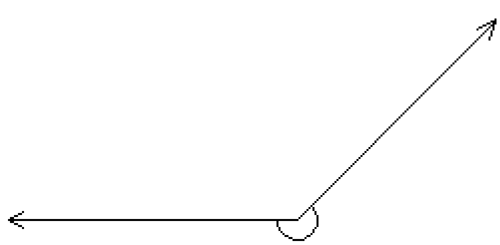
Reflexive Angle: between $180°$ and $360°$

Exercises

Identify each angle shown below as acute, right, obtuse, straight, or reflexive.

1. 

2. 

3. 

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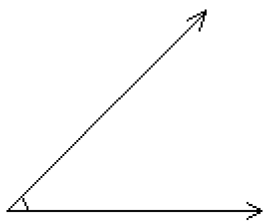


5.

Lines that form a $90°$ angle are called ***perpendicular***. As shown below, the needle should be perpendicular to the body surface for an intramuscular injection.

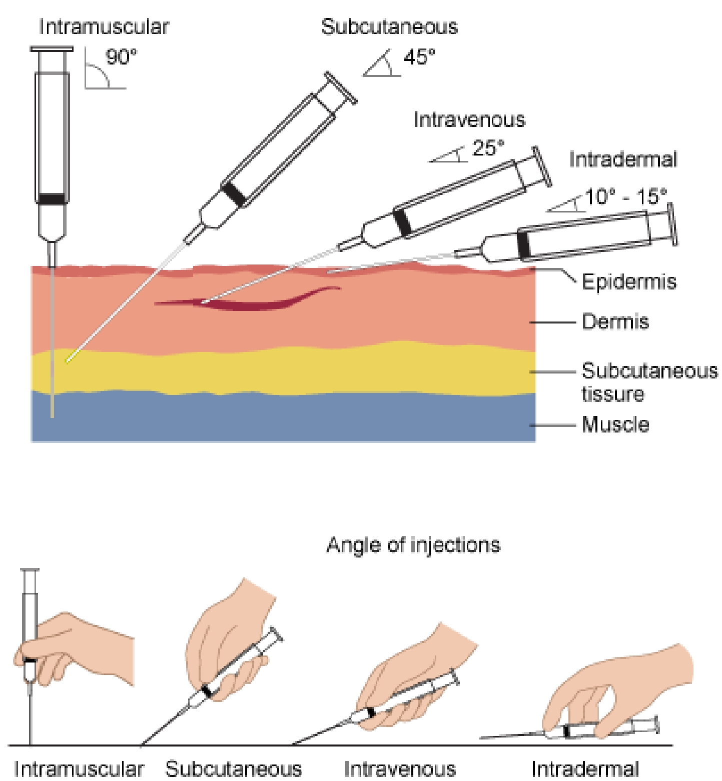
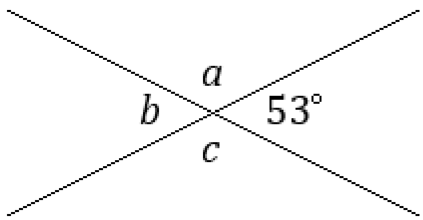


Image by British Columbia Institute of Technology (BCIT) on Wikimedia Commons

When lines cross, they form angles. No surprises there. If we know the measure of one angle, we may be able to determine the measures of the remaining angles using a little logic.

Exercises

Find the measure of each unknown angle.

6. 

Angles in Triangles

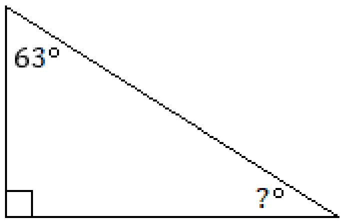
If you need to find the measures of the angles in a triangle, there are a few rules that can help.

The sum of the angles of every triangle is $180°$.

If any sides of a triangle have equal lengths, then the angles opposite those sides will have equal measures.

Exercises

Find the measures of the unknown angles in each triangle.



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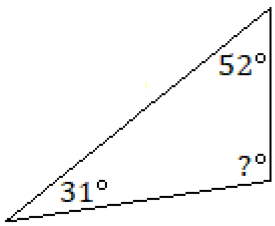

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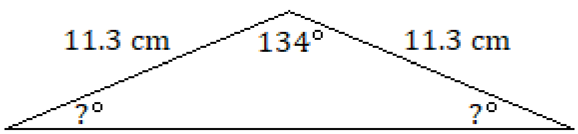

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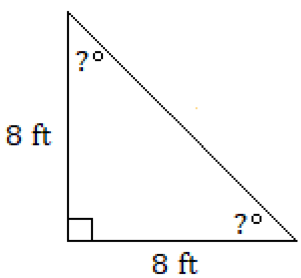

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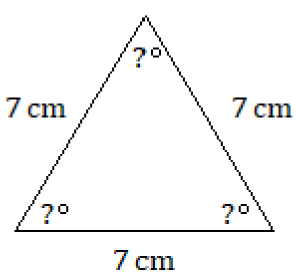
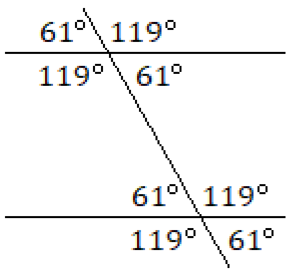


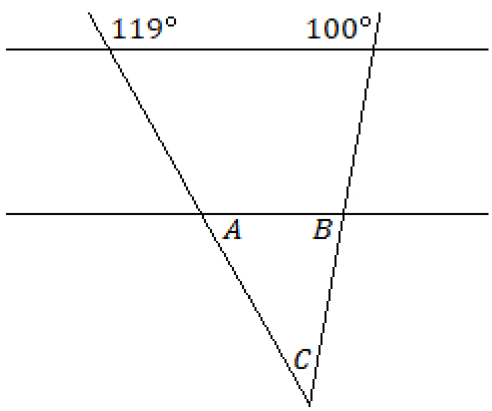
11.

Angles and Parallel Lines

Two lines that point in the exact same direction and will never cross are called ***parallel*** lines. If two parallel lines are crossed by a third line, sets of equally-sized angles will be formed, as shown in the following diagram. All four acute angles will be equal in measure, all four obtuse angles will be equal in measure, and any acute angle and obtuse angle will have a combined measure of $180°$.



Exercises



12. Find the measures of angles A, B, and C.

Degrees, Minutes, Seconds

It is possible to have angle measures that are not a whole number of degrees. It is common to use decimals in these situations, but the older method—which is called the ***degrees-minutes-seconds*** or ***DMS*** system—divides a degree using fractions out of 60: a minute is $\frac{1}{60}$ of a degree, and a second is $\frac{1}{60}$ of a minute, which means a second is $\frac{1}{3{,}600}$ of a degree. (Fortunately, these units work exactly like time; think of 1 degree as 1 hour.) For example, $2.5° = 2°30'$.

We will look at the procedure for converting between systems, but there are online calculators such as the one at https://www.fcc.gov/media/radio/dms-decimal which will do the conversions for you.

If you have latitude and longitude in DMS, like N $18°54'40''$ W $155°40'51''$, and need to convert it to decimal degrees, the process is fairly simple with a calculator.

Converting from DMS to Decimal Degrees

Enter $\text{degrees} + \text{minutes} \div 60 + \text{seconds} \div 3600$ in your calculator. Round the result to the fourth decimal place, if necessary.[1]

Exercises

Convert each angle measurement from degrees-minutes-seconds into decimal form. Round to the nearest ten-thousandth, if necessary.

13. $18°54'40''$

14. $155°40'51''$

15. $34°11'32.5''$

Going from decimal degrees to DMS is a more complicated process.

Converting from Decimal Degrees to DMS

1. The whole-number part of the angle measurement gives the number of degrees.
2. Multiply the decimal part by 60. The whole number part of this result is the number of minutes.
3. Multiply the decimal part *of the minutes* by 60. This gives the number of seconds (including any decimal part of seconds).

For example, let's convert $15.3740°$.

1. The *degrees* part of our answer will be 15.
2. The decimal part times 60 is $0.3740 \cdot 60 = 22.44$ minutes. The *minutes* part of our answer will be 22.
3. The decimal part times 60 is $0.44 \cdot 60 = 26.4$ seconds. The *seconds* part of our answer will be 26.4.

So $15.3740° = 15°22'26.4''$.

Exercises

Convert each angle measurement from decimal into degrees-minutes-seconds form.

16. $29.9750°$

17. $31.1375°$

18. $76.3467°$

Exercise Answers

Notes

1. We round to four decimal places because 1 second of angle is $\frac{1}{3,600}$ of a degree. This is a smaller fraction than $\frac{1}{1,000}$ so our precision is slightly better than the thousandths place.

[18]

Triangles

Triangular supports hold a boat in the air while it undergoes repairs at the Front Street Shipyard in Belfast, Maine.

Classifying Triangles

We can classify triangles into three categories based on the lengths of their sides.

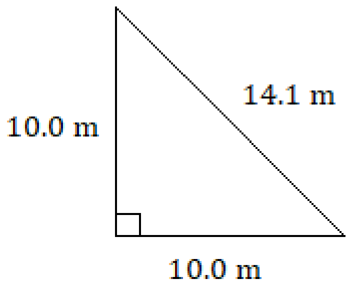
- Equilateral triangle: all three sides have the same length
- Isosceles triangle: exactly two sides have the same length
- Scalene triangle: all three sides have different lengths

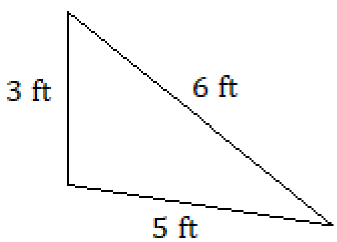
We can also classify triangles into three categories based on the measures of their angles.

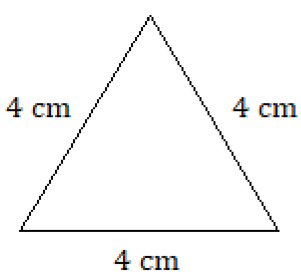
- Obtuse triangle: one of the angles is an obtuse angle
- Right triangle: one of the angles is a right angle
- Acute triangle: all three of the angles are acute

Exercises

Classify each triangle by angle and side. For example, “acute scalene”.)

1. 

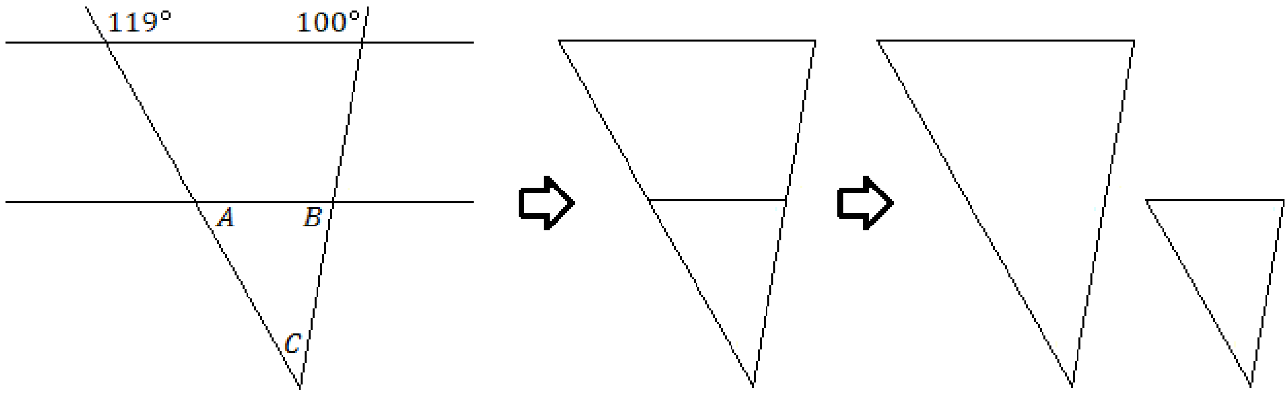
2. 

3. 

You may use a calculator for the remainder of this module.

Similar Triangles

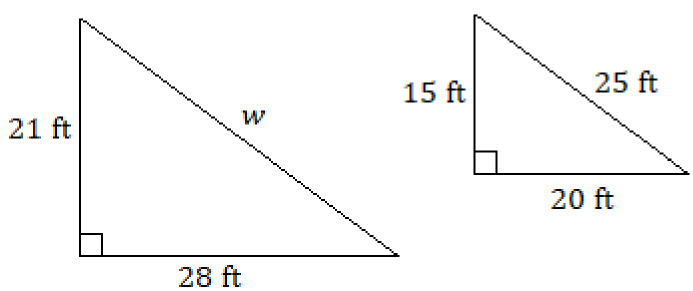
In one of the diagrams in Module 17, the parallel lines included two similar triangles, although they may be hard to see.



Two triangles are ***similar*** if the three angles of one triangle have the same measure as the three angles of the second triangle. The lengths of the sides of similar triangles will be in the same proportion. The triangles will have the same shape but the lengths will be scaled up or down.

Exercises

Assume that each pair of triangles are similar. Use a proportion to find each unknown length.



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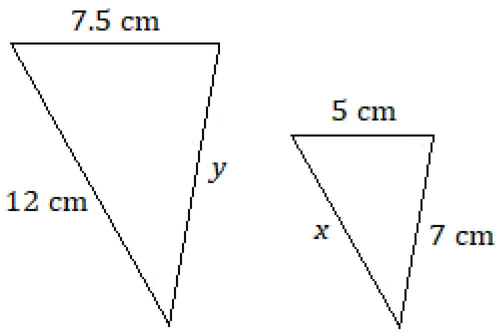
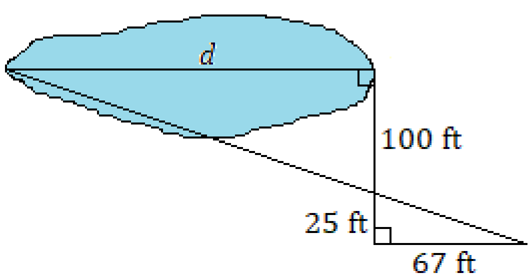


5.

Recognizing corresponding sides can be more difficult when the figures are oriented differently.

Exercises

Assume that each pair of triangles are similar. Use a proportion to find each unknown length.



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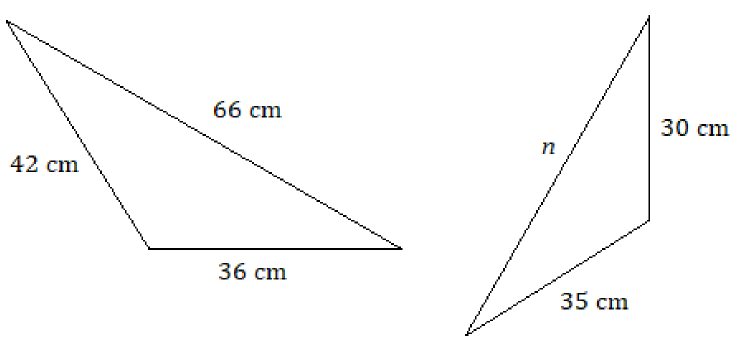
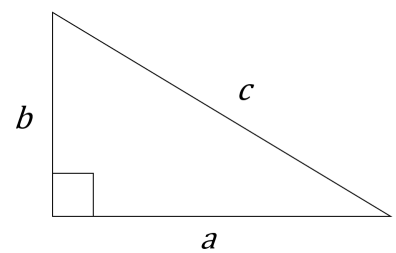


7.

The Pythagorean Theorem

In a right triangle, the two sides that form the right angle are called the ***legs***. The side opposite the right angle, which will always be the longest side, is called the ***hypotenuse***.

The ***Pythagorean theorem*** says that the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse.



The Pythagorean Theorem

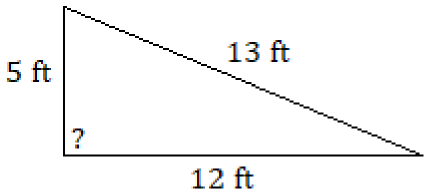
In a right triangle with legs a and b and hypotenuse c,

$a^2 + b^2 = c^2$

If you know the lengths of all three sides of a triangle, you can use the Pythagorean theorem to verify whether the triangle is a right triangle or not. The ancient Egyptians used this method for surveying when they needed to redraw boundaries after the yearly flooding of the Nile washed away their previous markings.[1]

Exercises

Use the Pythagorean theorem to determine whether either of the following triangles is a right triangle.



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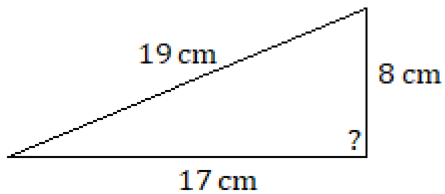


9.

Before we continue, we need to briefly discuss square roots. Calculating a square root is the opposite of squaring a number. For example, $\sqrt{49}=7$ because $7^2=49$. If the number under the square root symbol is not a perfect square like 49, then the square root will be an irrational decimal that we will round off as necessary.

Exercises

Use a calculator to find the value of each square root. Round to the hundredths place.

10. $\sqrt{50}$

11. $\sqrt{296}$

12. $\sqrt{943}$

We most often use the Pythagorean theorem to calculate the length of a missing side of a right triangle. Here are three different versions of the Pythagorean theorem arranged to find a missing side, so you don't have to use algebra with $a^2+b^2=c^2$.

The Pythagorean Theorem, three other versions

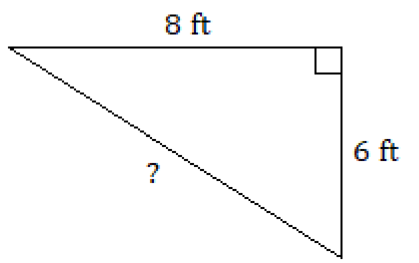
$$c = \sqrt{a^2 + b^2}$$

$$b = \sqrt{c^2 - a^2}$$

$$a = \sqrt{c^2 - b^2}$$

Exercises

Find the length of the missing side for each of these right triangles. Round to the nearest tenth, if necessary.



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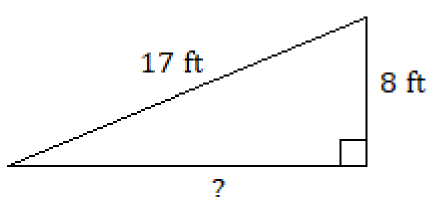

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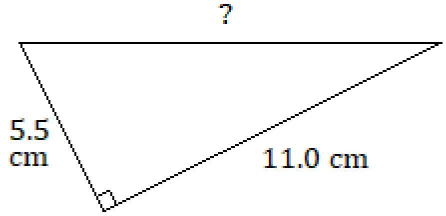

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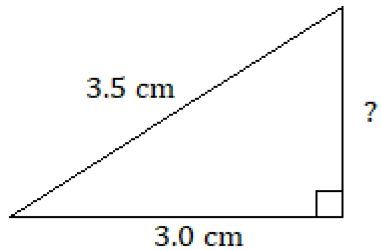


16.

Exercise Answers

Notes

1. The surveyors were called "rope-stretchers" because they used a loop of rope 12 units long with 12 equally-spaced knots. Three rope-stretchers each held a knot, forming a triangle with lengths 3, 4, and 5 units. When the rope was stretched tight, they knew that the angle between the 3-unit and 4-unit sides was a right angle because $3^2 + 4^2 = 5^2$. From *Discovering Geometry: an Inductive Approach* by Michael Serra, Key Curriculum Press, 1997.

[19]

Area of Polygons and Circles

You may use a calculator in this module as needed.

We have seen that the ***perimeter*** of a polygon is the distance around the outside. Perimeter is a length, which is one-dimensional, and so it is measured in linear units (feet, centimeters, miles, etc.). The ***area*** of a polygon is the amount of two-dimensional space inside the polygon, and it is measured in square units: square feet, square centimeters, square miles, etc.

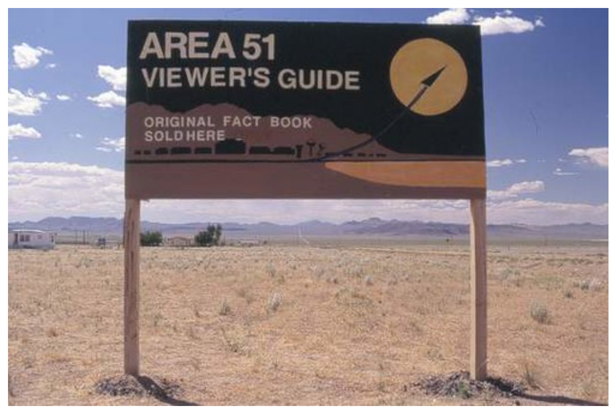


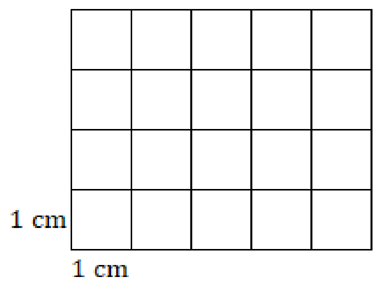
Photo by Paolo Macorig on flickr.

You can always think of area as the number of squares required to completely fill in the shape.

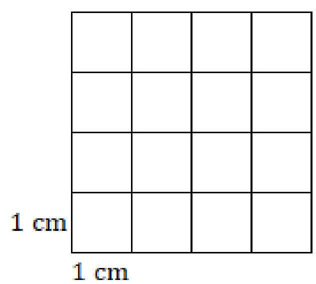
Area: Rectangles and Squares

Exercises

1. Find the area of this rectangle.



2. Find the area of this square.



There are of course formulas for finding the areas of rectangles and squares; we don't have to count little squares.

Area of a Rectangle

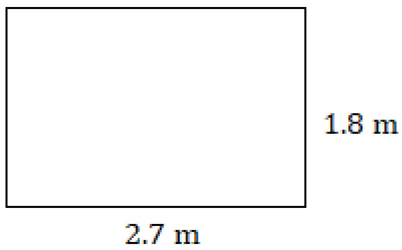
$A = lw$[1] or $A = bh$

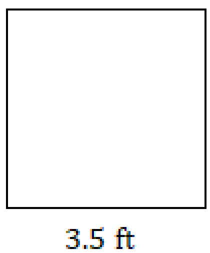
Area of a Square

$A = s^2$

Exercises

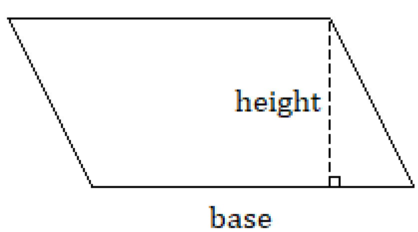
Find the area of each figure.

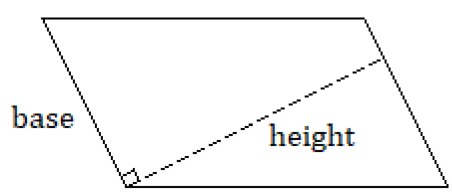
3. 

4. 

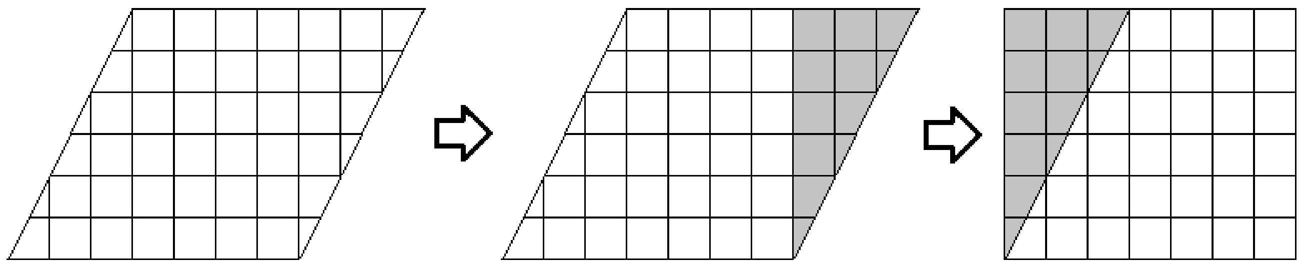
Area: Parallelograms

Another common polygon is the ***parallelogram***, which looks like a tilted rectangle. As the name implies, the pairs of opposite sides are parallel and have the same length. Notice that, if we label one side as a base of the parallelogram, we have a perpendicular height which is *not* the length of the other sides.





The following set of diagrams shows that we can cut off part of a parallelogram and rearrange the pieces into a rectangle with the same base and height as the original parallelogram. A parallelogram with a base of 7 units and a vertical height of 6 units is transformed into a 7 by 6 rectangle, with an area of 42 square units.



Therefore, the formula for the area of a parallelogram is identical to the formula for the area of a rectangle, provided that we are careful to use the base and the height, which must be perpendicular.

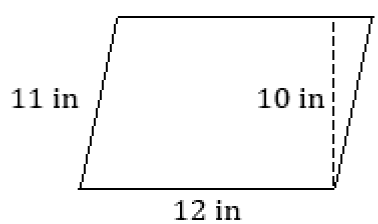
Area of a Parallelogram

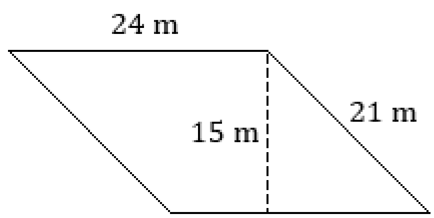
$A = bh$

Exercises

Find the area of each parallelogram.

5.

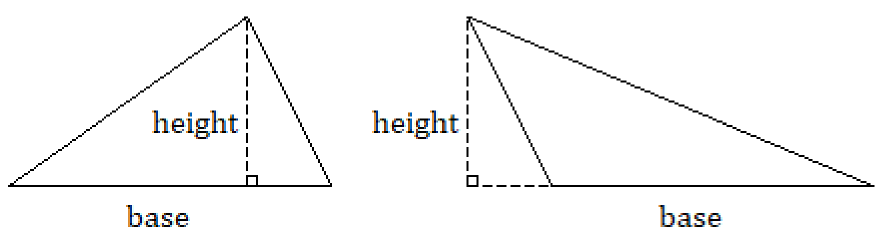




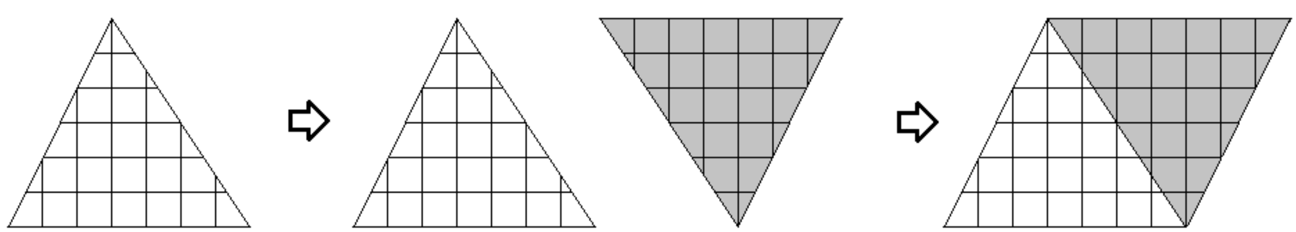
6.

Area: Triangles

When we are finding the area of a triangle, we need to identify a base and a height that is perpendicular to that base. If the triangle is obtuse, you may have to imagine the height outside of the triangle and extend the base line to meet it.



As shown below, any triangle can be doubled to form a parallelogram. Therefore, the area of a triangle is one half the area of a parallelogram with the same base and height.



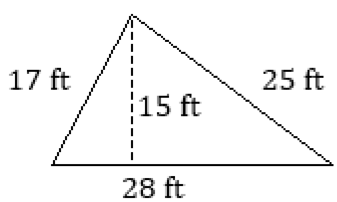
Area of a Triangle

$A = \frac{1}{2}bh$ or $A = bh \div 2$

As with a parallelogram, remember that the height must be perpendicular to the base.

Exercises

Find the area of each triangle.



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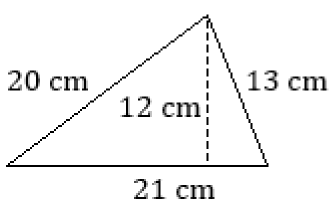

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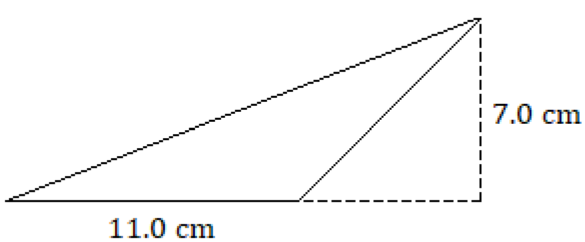
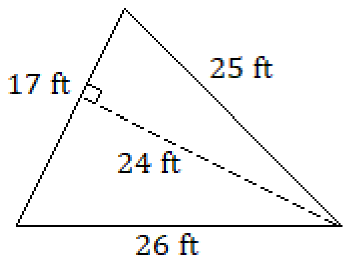
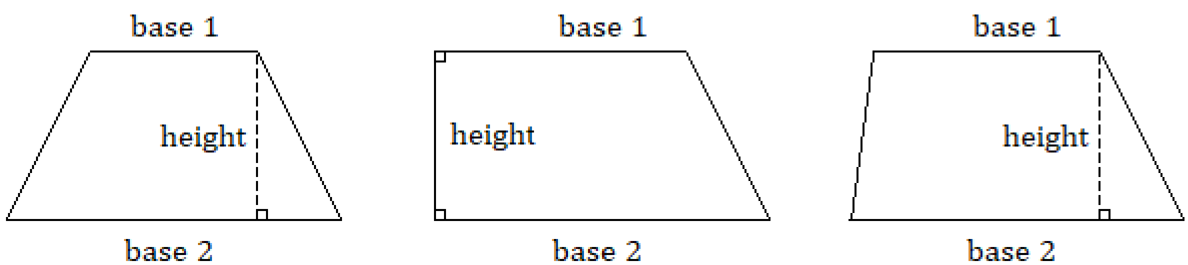


9.

10. 

Area: Trapezoids

A somewhat less common quadrilateral is the ***trapezoid***, which has exactly one pair of parallel sides, which we call the bases. The first example shown below is called an isosceles trapezoid because, like an isosceles triangle, its two nonparallel sides have equal lengths.



There are a number of ways to show where the area formula comes from, but the explanations are better in video because they can be animated.[234]

Area of a Trapezoid

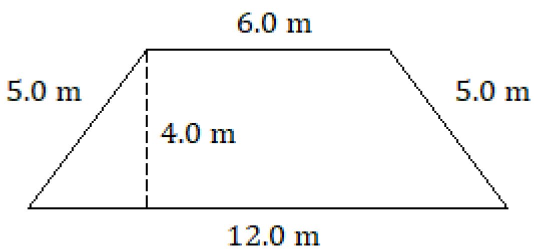
$A = \frac{1}{2}h(b_1 + b_2)$ or $A = (b_1 + b_2)h \div 2$

Don't be intimidated by the subscripts on b_1 and b_2; it's just a way to name two different measurements using the same letter for the variable. (Many people call the bases a and b instead; feel free to write it whichever way you prefer.) Whatever you call them, you just add the two bases, multiply by the height, and take half of that.

Exercises

11. The soaking tubs at Hot Lake Springs in La Grande, Oregon have benches formed by seven trapezoids. Each bench has an outer edge 33 inches long, an inner edge 23 inches long, and the edges are 11 inches apart. How much seating area do the benches provide? (Ignore the gaps between the boards.)

Find the area of each trapezoid.



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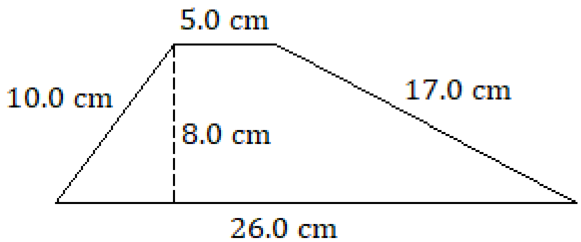

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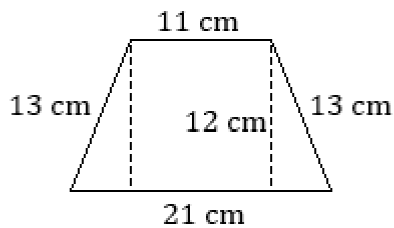
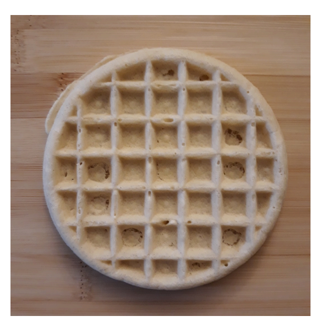


14.

Rocky the cat lounges at Hot Lake Springs.

Area: Circles



The area of a circle is π times the square of the radius: $A = \pi r^2$. The units are still square units, even though a circle is round. (Think of the squares on a round waffle.) Because we can't fit a whole number of squares—or an exact fraction of squares—inside the circle, the area of a circle will be an approximation.

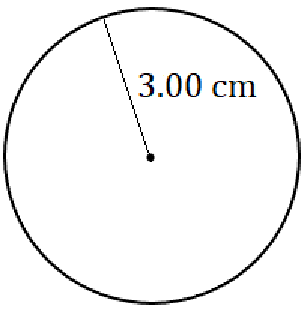
If your calculator doesn't have a π key, use the approximation $\pi \approx 3.1416$.

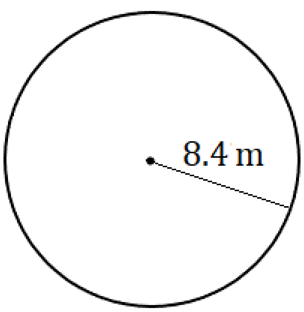
Area of a Circle

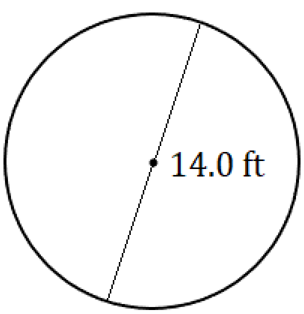
$A = \pi r^2$

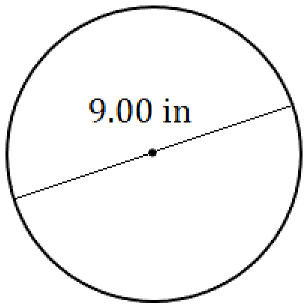
Exercises

Find the area of each circle. Round each answer to the appropriate number of significant figures.

15. 

16. 

17. 



18.

Exercise Answers

Notes

1. You might choose to use capital letters for the variables here because a lowercase letter "l" can easily be mistaken for a number "1".
2. https://youtu.be/yTnYRpcZA9c
3. https://youtu.be/WZtO3oERges
4. https://youtu.be/uLHc6Br2veg

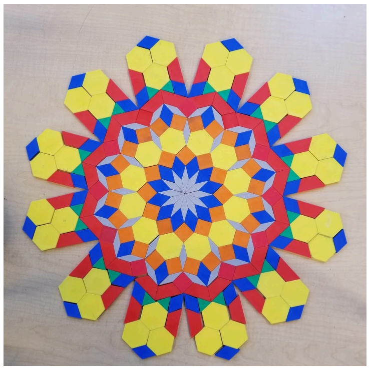
[20]

Composite Figures

You may use a calculator in this module as needed.

Many objects have odd shapes made up of simpler shapes. A ***composite figure*** is a geometric figure which is formed by—or *composed* of—two or more basic geometric figures.

We will look at a handful of fairly simple examples, but this concept can of course be extended to much more complicated figures.



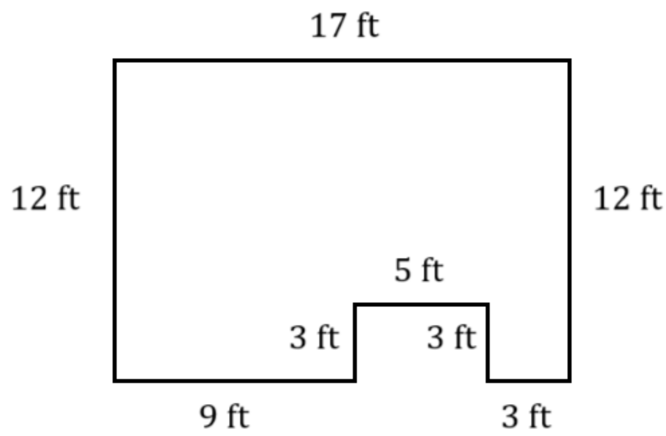
This sunburst design is composed of 276 tiles.

Composite Figures with Polygons

To find the area of a composite figure, it is generally a good idea to divide it into simpler shapes—rectangles, triangles, etc.—and either add or subtract their areas as necessary. You may need to figure out some unmarked dimensions during the process.

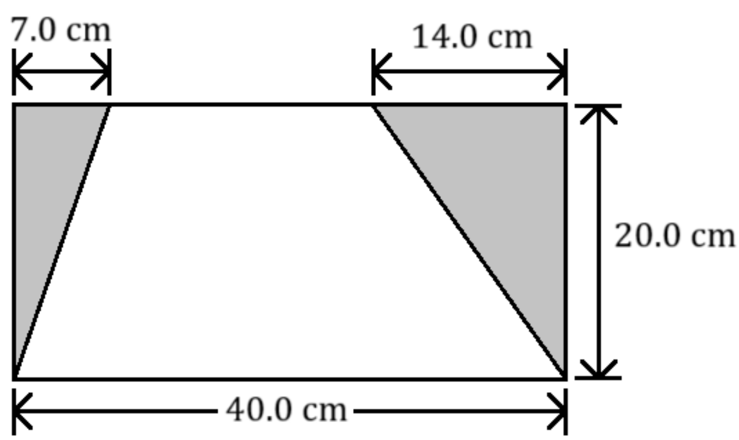
Exercises

A floor plan of a room is shown. The room is a 12-foot by 17-foot rectangle, with a 3-foot by 5-foot rectangle cut out of the south side.



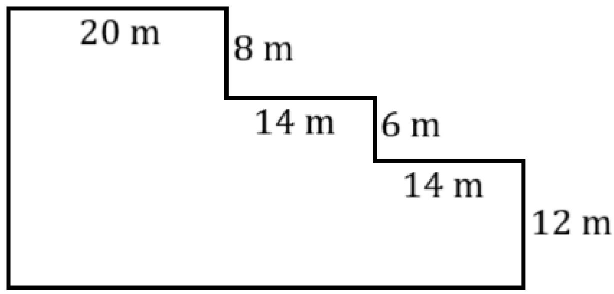
1. Determine the amount of molding required for the perimeter of the room.
2. Determine the amount of flooring required to cover the entire area.

A trapezoid is formed by removing two right triangles from a rectangle, as shown.



3. Determine the area of the trapezoid *without* using the trapezoid formula from Module 19.
4. Determine the area of the trapezoid using the trapezoid formula from Module 19. Does this agree with your previous answer?

A plan for an irregular parking lot is shown.



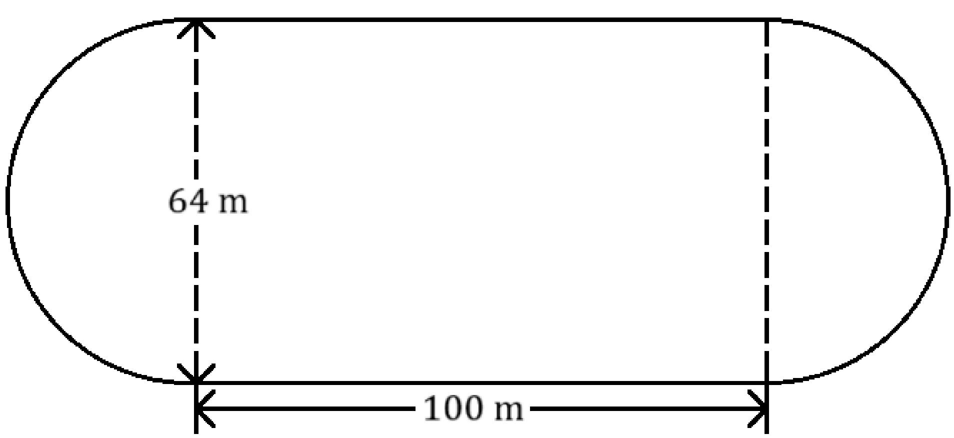
5. Calculate the perimeter.
6. Calculate the area.

Composite Figures with Circles

If a composite figure includes parts of circles, you'll need to approximate you answer because your calculation will have π in it.

Exercises

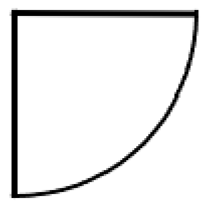
A high school is building a track at its athletic fields. The track, which is formed by two straight sides and two semicircles as shown in the plans below, is supposed to have a total length of 400 meters.



7. Determine the distance around the track. Will the track be the right length?

8. After the new track is built, landscapers need to lay sod on the field inside the track. What is the area of the field inside the track?

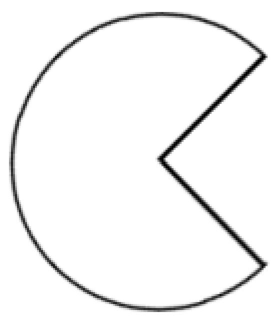
The radius of the quarter circle is 50.0 centimeters.



9. Calculate the area.

10. Calculate the perimeter.

A quarter circle has been removed from a circle with a diameter of 7.0 feet.

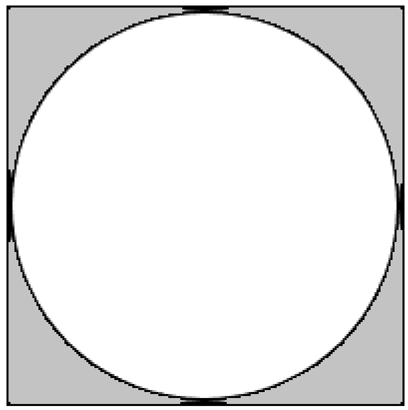


11. Calculate the area.

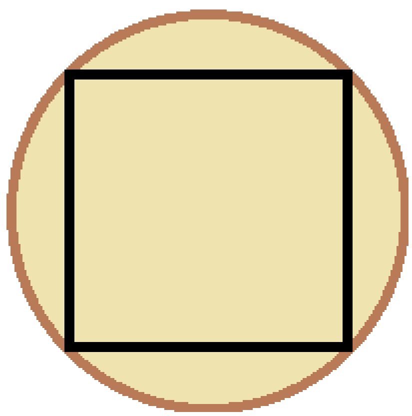
12. Calculate the perimeter.

If we need to determine a fraction or percent out of the whole, we may be able to solve the problem without knowing any actual measurements.

Exercises



13. Circular disks are being cut from squares of sheet metal, with the remainder around the corners being discarded. Assuming that the circles are made as large as possible, what percent of the sheet metal will be discarded?



14. A log with a circular cross-section is being planed to create a beam with a square cross-section. Assuming that the beam is made as large as possible, what percent of the log will be removed?

Exercise Answers

[21]

Converting Units of Area

You may use a calculator throughout this module.

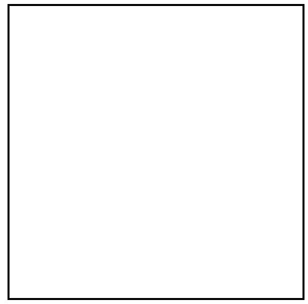
When we are converting between units of area, we need to be aware that square units behave differently than linear units. We'll begin by using our knowledge of linear units to investigate how area conversions are different.

U.S. System: Converting Measurements of Area

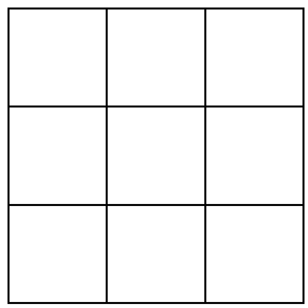
Exercises

1. A hallway's floor is 9 yards long and 2 yards wide. Convert each measurement to feet, then determine the area of the floor in square feet.
2. A hallway's floor has an area of 18 square yards. Determine the area of the floor in square feet.
3. A tabletop measures 24.0 inches by 42.0 inches. Convert each measurement to feet, then determine the area of the tabletop in square feet.
4. A tabletop has an area of $1,008$ square inches. Determine the area of the tabletop in square feet.

It may help to visualize the relationship between square yards and square feet. Consider a square yard; the area of a square with sides 1 yard long.



We know that 1 yard = 3 feet, so we can divide the square into three sections vertically and three sections horizontally to convert both dimensions of the square from yards to feet.



This forms a 3 by 3 grid, which shows us visually that 1 square yard equals 9 square feet! Instead of 1 to 3, the conversion ratio for the areas is 1 to 3^2, or 1 to 9.

Here's another way to think about it without a diagram: $1 \text{ yd} = 3 \text{ ft}$, so $(1 \text{ yd})^2 = (3 \text{ ft})^2$. To remove the parentheses, we must square the number *and* square the units: $(3 \text{ ft})^2 = 3^2 \text{ ft}^2 = 9 \text{ ft}^2$.

More generally, we need to **square** the linear conversion factors when converting units of area. If the linear units have a ratio of 1 to n, the square units will have a ratio of 1 to n^2.

Here are the conversion ratios for area in the U.S. system. As always, if you discover other conversion ratios that aren't provided here, it would be a good idea to write them down so you can use them as needed.

$$1 \text{ ft}^2 = 144 \text{ in}^2$$
$$1 \text{ yd}^2 = 9 \text{ ft}^2$$
$$1 \text{ yd}^2 = 1,296 \text{ in}^2$$
$$1 \text{ acre (ac)} = 4,840 \text{ yd}^2$$
$$1 \text{ ac} = 43,560 \text{ ft}^2$$
$$1 \text{ mi}^2 = 640 \text{ ac}$$
$$1 \text{ mi}^2 = 3,097,600 \text{ yd}^2$$
$$1 \text{ mi}^2 = 27,878,400 \text{ ft}^2$$

If you're curious, an acre is defined as the area of a 660 foot by 66 foot rectangle. (That's a furlong by a chain. Why? Because that's the amount of land that a medieval farmer with a team of eight oxen could plow in one day.)[1] An acre is defined as a unit of area; it would be wrong to say "acres squared" or put an exponent of 2 on the units.

Exercises

5. A rectangular sheet of fabric has an area of 6 square yards. Find its area in square inches.

6. A proposed site for an elementary school is 600 feet by 600 feet. Determine its area in acres, rounded to the nearest tenth.

Metric System: Converting Measurements of Area

Now let's take a look at converting area using the metric system.

Exercises

7. A hallway's floor is 9 meters long and 2 meters wide. Convert each measurement to centimeters, then determine the area of the floor in square centimeters.
8. A hallway's floor has an area of 18 square meters. Determine the area of the floor in square centimeters.
9. A sheet of A4 paper measures 210 mm by 297 mm. Convert each measurement to centimeters, then determine the area in square centimeters.
10. A sheet of A4 paper has an area of $62,370$ square millimeters. Determine the area in square centimeters.

As shown below, the conversions in the metric system are all powers of ten, which means they are all about moving the decimal point.

$$1 \text{ cm}^2 = 100 \text{ mm}^2$$

$$1 \text{ m}^2 = 10,000 \text{ cm}^2$$

$$1 \text{ m}^2 = 1,000,000 \text{ mm}^2$$

$$1 \text{ hectare (ha)} = 10,000 \text{ m}^2$$

$$1 \text{ km}^2 = 1,000,000 \text{ m}^2$$

$$1 \text{ km}^2 = 100 \text{ ha}$$

A hectare is defined as a square with sides 100 meters long. Dividing a square kilometer into ten rows and ten columns will make a 10 by 10 grid of 100 hectares. As with acres, it would be wrong to say "hectares squared" or put an exponent of 2 on the units.

Exercises

11. A rectangular sheet of fabric has an area of $60,000$ square centimeters. Find its area in square meters.

12. A proposed site for an elementary school is 200 meters by 200 meters. Find its area, in hectares.

Both Systems: Converting Measurements of Area

Converting between the U.S. and metric systems will involve messy decimal values. For example, because $1 \text{ in} = 2.54 \text{ cm}$, we can square both numbers and find that $(1 \text{ in})^2 = (2.54 \text{ cm})^2 = 6.4516 \text{ cm}^2$. The conversions are rounded to five significant figures in the table below.

$1 \text{ in}^2 = 6.4516 \text{ cm}^2$

$1 \text{ m}^2 \approx 10.764 \text{ ft}^2$

$1 \text{ m}^2 \approx 1.1960 \text{ yd}^2$

$1 \text{ mi}^2 \approx 2.5900 \text{ km}^2$

$1 \text{ ha} \approx 2.4711 \text{ ac}$ or $1 \text{ ac} \approx 0.40469 \text{ ha}$

Exercises

13. The area of Portland, Oregon is $145\ \mathrm{mi}^2$. Convert this area to square kilometers.

14. How many hectares is a $5,\overline{0}00$ acre ranch?

15. A standard sheet of paper measures 8.50 inches by 11.00 inches. What is the area in square centimeters?

16. A football pitch (soccer field) is 100.0 meters long and 70.0 meters wide. What is its area in square feet?

Before the Denmark v Germany match, Rotterdam, Women's Euro 2017.

Areas of Similar Figures

Earlier in this module, it was stated that if the linear units have a ratio of 1 to n, the square units will have a ratio of 1 to n^2. This applies to similar figures as well.

> If the linear dimensions of two similar figures have a ratio of 1 to n, then the areas will have a ratio of 1 to n^2.

This is true for circles, similar triangles, similar rectangles, similar hexagons, you name it. We'll verify this in the following exercises.

Exercises

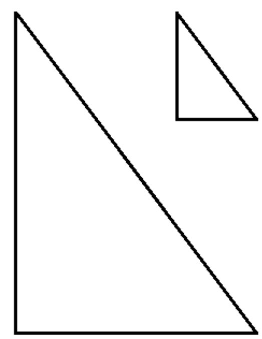
A personal pizza has a 7-inch diameter. A medium pizza has a diameter twice that of a personal pizza.

17. Determine the area of the medium pizza.

18. Determine the area of the personal pizza.

19. What is the ratio of the areas of the two pizzas?

A right triangle has legs 3 cm and 4 cm long. A larger right triangle has legs triple these dimensions.



20. Determine the area of the larger triangle.

21. Determine the area of the smaller triangle.

22. What is the ratio of the areas of the two triangles?

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/Acre

[22]

Surface Area of Common Solids

You may use a calculator throughout this module.

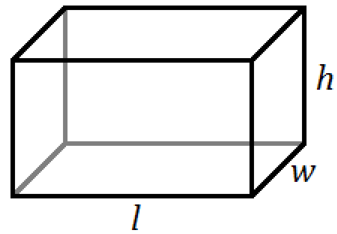
We will now turn our attention from two-dimensional figures to three-dimensional figures, which we often call ***solids***, even if they are hollow inside.

In this module, we will look the surface areas of some common solids. (We will look at volume in a later module.) Surface area is what it sounds like: it's the sum of the areas of all of the outer surfaces of the solid. When you are struggling to wrap a present because your sheet of wrapping paper isn't quite big enough, you are dealing with surface area.

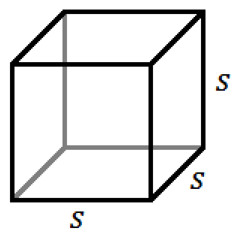
There are two different kinds of surface area that are important: the ***lateral surface area*** (LSA) and ***total surface area*** (TSA).

To visualize the difference between LSA and TSA, consider a can of food. The lateral surface area would be used to measure the size of the paper label around the can. The total surface area would be used to measure the amount of sheet metal needed to make the can. In other words, the total surface area includes the top and bottom, whereas the lateral surface area does not.

Surface Area: Rectangular Solids



A ***rectangular solid*** looks like a rectangular box. It has three pairs of equally sized rectangles on the front and back, on the left and right, and on the top and bottom.



A ***cube*** is a special rectangular solid with equally-sized squares for all six faces.

The lateral surface area is the combined total area of the four vertical faces of the solid, but not the top and bottom. If you were painting the four walls of a room, you would be thinking about the lateral surface area.

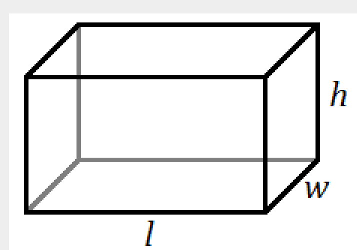
The total surface area is the combined total area of all six faces of the solid. If you were painting the four walls, the floor, and the ceiling of a room, you would be thinking about the total surface area.

For a rectangular solid with length l, width w, and height h...

$$LSA = 2lh + 2wh$$

$$TSA = 2lh + 2wh + 2lw$$

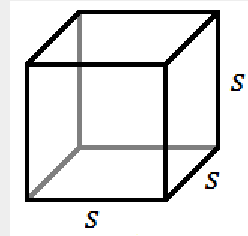
$$TSA = LSA + 2lw$$



For a cube with side length s...

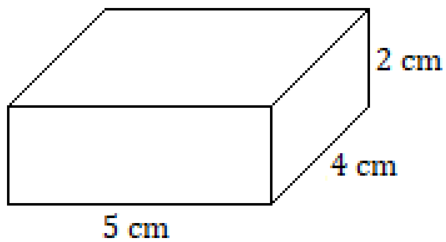
$LSA = 4s^2$

$TSA = 6s^2$



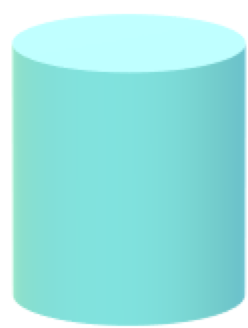
Note: These dimensions are sometimes called base, depth, and height.

Exercises



1. Find the lateral surface area of this rectangular solid.
2. Find the total surface area of this rectangular solid.

Surface Area: Cylinders



As mentioned earlier in this module, the lateral surface area of a soup can is the paper label, which is a rectangle. Therefore, the lateral surface area of a ***cylinder*** is a rectangle; its width is equal to the circumference of the circle, $2\pi r$, and its height is the height of the cylinder.

Since a cylinder has equal-sized circles at the top and bottom, its total surface area is equal to the lateral surface area plus twice the area of one of the circles.

For a cylinder with radius r and height h...

$LSA = 2\pi rh$

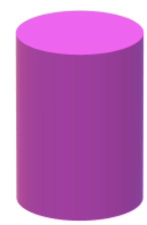
$TSA = 2\pi rh + 2\pi r^2$

$TSA = LSA + 2\pi r^2$

Be aware that if you are given the diameter of the cylinder, you will need to cut it in half before using these formulas.

Exercises

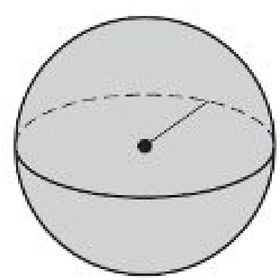
A cylinder has a diameter of $10.0\ \text{cm}$ and a height of $15.0\ \text{cm}$.



3. Find the lateral surface area.

4. Find the total surface area.

Surface Area: Spheres



The final solid of this module is the ***sphere***, which can be thought of as a circle in three dimensions: every point on the surface of a sphere is the same distance from the center. Because of this, a sphere has only one important measurement: its radius. Of course, its diameter could be important also, but the idea is that a sphere doesn't have different dimensions such as length, width, and height. A sphere has the same radius (or diameter) in every direction.

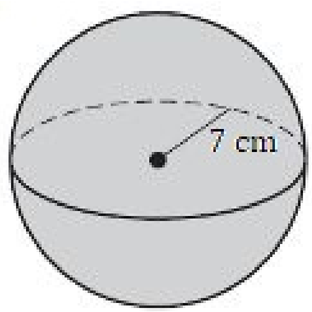
We would need to use calculus to derive the formula for the surface area of a sphere, so we'll just assume it's true and get on with the business at hand. Notice that, because a sphere doesn't have top or bottom faces, we don't need to worry about finding the lateral surface area. The only surface area is the total surface area.

For a sphere with radius r or diameter d...

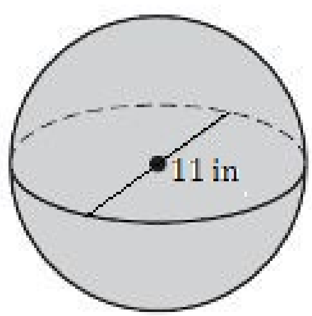
$SA = 4\pi r^2$ or $SA = \pi d^2$

Coincidentally, the surface area of a sphere is 4 times the area of the cross-sectional circle at the sphere's widest part. You may find it interesting to try to visualize this, or head to the kitchen for a demonstration: if you cut an orange into four quarters, the peel on one of those quarter oranges has the same area as the circle formed by the first cut.

Exercises



5. Find the surface area of this sphere.



6. Find the surface area of this sphere.

Exercise Answers

[23]

Area of Regular Polygons

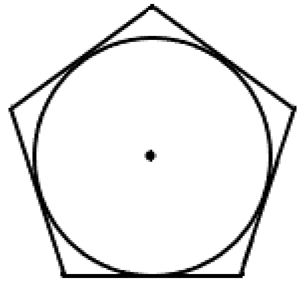
You may use a calculator throughout this module.

"The Pentagon" by David B. Gleason is licensed under CC BY-SA 2.0

The Pentagon building spans 28.7 acres $(116,000 \text{ m}^2)$, and includes an additional 5.1 acres $(21,000 \text{ m}^2)$ as a central courtyard.[1] (That's roughly 25 American football fields… or 23 Canadian football fields.) In this module, we will focus on calculating the area of regular polygons such as this one.

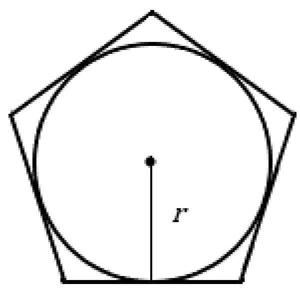
A ***regular polygon*** has all sides of equal length and all angles of equal measure. Because of this symmetry, a circle can be ***inscribed***—drawn inside the polygon touching each side at one point—or ***circumscribed***—drawn outside the polygon intersecting each vertex. We'll focus on the inscribed circle first.

Using the Distance from the Center to an Edge



inscribed circle

Let's call the radius of the inscribed circle lowercase r; this is the distance from the center of the polygon perpendicular to one of the sides. (The inner radius is more commonly called the ***apothem*** and labeled a, but we are trying to keep the jargon to a minimum in this textbook.)



We can derive a formula for the area of a regular polygon by dividing the polygon into equally-sized triangles and combining the areas of those triangles. If the length of each side of is s, each triangle's area is its base times its height divided by 2, or $s \cdot r \div 2$. If the polygon has n sides, then there are n of these triangles, and the total combined area is $n \cdot s \cdot r \div 2$.

Area of a Regular Polygon (with a radius drawn to the center of one side)

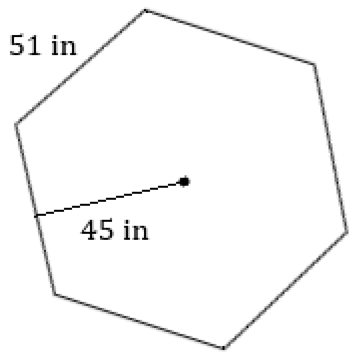
For a regular polygon with n sides of length s, and inscribed (inner) radius r,

$A = nsr \div 2$

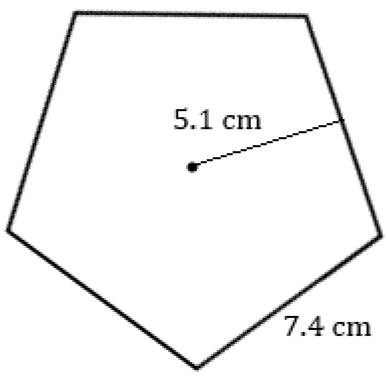
Note: This formula is more commonly written as one-half the apothem times the perimeter: $A = \frac{1}{2}ap$. The apothem is what we're calling r, and the perimeter is $n \cdot s$, so $A = \frac{1}{2}ap$ and $A = nsr \div 2$ are equivalent formulas.

Exercises

1. Calculate the area of this regular hexagon.



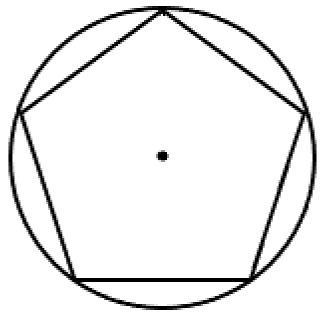
2. Calculate the area of this regular pentagon.



3. A stop sign has a height of 30 inches, and each edge measures 12.5 inches. Find the area of the sign.

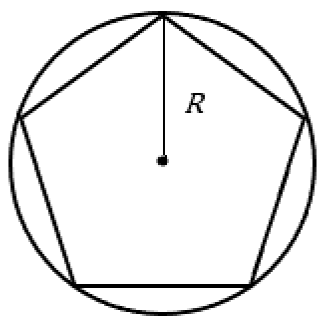
Using the Distance from the Center to a Vertex

Okay, but what if we know the distance from the center to a vertex (a corner of the polygon) instead of the distance from the center to an edge? We'll need to imagine a circumscribed circle.



circumscribed circle

Let's call the radius of the circumscribed circle capital R; this is the distance from the center of the polygon to one of the vertices (corners).



Area of a Regular Polygon (with a radius drawn to a vertex)

For a regular polygon with n sides of length s, and circumscribed (outer) radius R,

$A = 0.25ns\sqrt{4R^2 - s^2}$

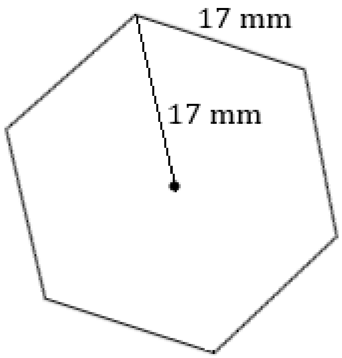
or

$A = ns\sqrt{4R^2 - s^2} \div 4$

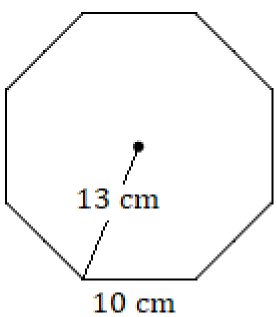
Your author created this formula because every other version of it uses trigonometry, which we haven't covered yet. [2]

Exercises

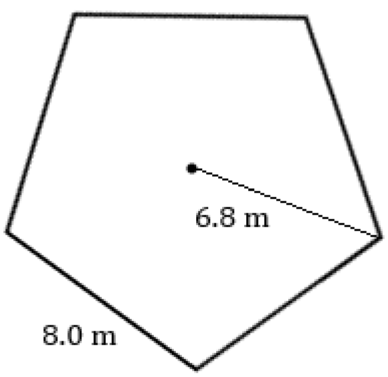
4. Calculate the area of this regular hexagon.



5. Calculate the area of this regular octagon.



6. Calculate the area of this regular pentagon.

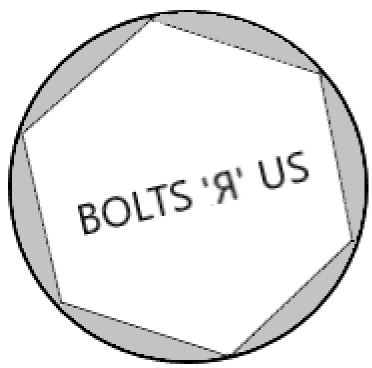


Fun fact: In a regular hexagon, the radius to a vertex is always equal to the side length. If you divide the hexagon into six equilateral triangles, you'll see why.

Composite Figures with Regular Polygons

As you know, a ***composite figure*** is a geometric figure which is formed by joining two or more basic geometric figures. Let's look at a composite figure formed by a circle and a regular polygon.

Exercises



7. The hexagonal head of a bolt fits snugly into a circular cap with a circular hole with inside diameter 46 mm as shown in this diagram. Opposite sides of the bolt head are 40 mm apart. Find the total empty area in the hole around the edges of the bolt head.

Exercise Answers

Notes

1. Source: https://en.wikipedia.org/wiki/The_Pentagon
2. This formula is derived from dividing the polygon into *n* equally-sized triangles and combining the areas of those triangles; it includes a square root because it involves the Pythagorean theorem.

[24]

Volume of Common Solids

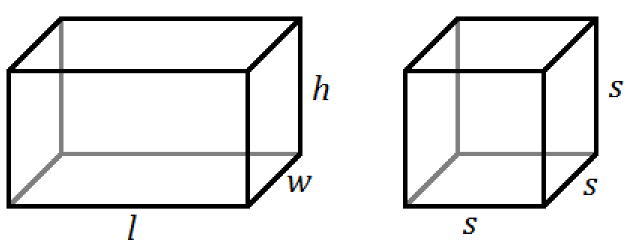
You may use a calculator throughout this module.

Note: Some of the diagrams in this module have dimensions with only one significant figure, but we would lose a lot of information if we rounded the results to only one sig fig. Therefore, we will not necessarily follow the accuracy-based rounding rules in the answer key for this module.

Wanda Ortiz of The Iron Maidens turns up the volume. Photo by Alejandro Páez on flickr.

The surface area of a solid is the sum of the areas of all its faces; therefore, surface area is two-dimensional and measured in square units. The ***volume*** is the amount of space inside the solid. Volume is three-dimensional, measured in cubic units. You can imagine the volume as the number of cubes required to completely fill up the solid.

Volume: Rectangular Solids



Volume of a Rectangular Solid

For a rectangular solid with length l, width w, and height h:

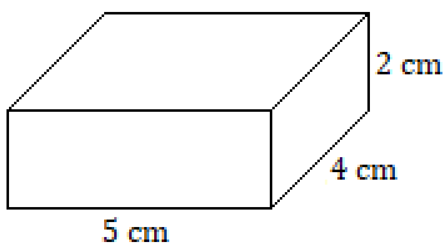
$$V = lwh$$

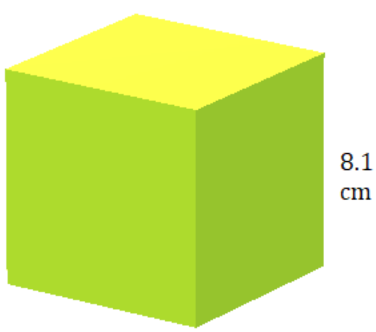
For a cube with side length s:

$$V = s^3$$

Exercises

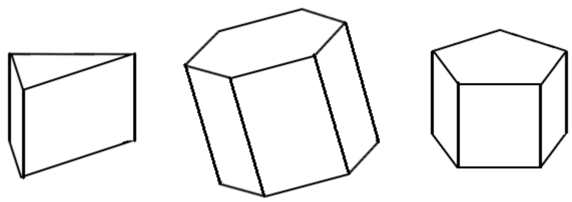
Find the volume of each solid.

1. 

2. 

Volume: Rectangular Prisms

A solid with two equal sized polygons as its bases and rectangular lateral faces is called a right-angle prism. Some examples are shown below. We will refer to them simply as ***prisms*** in this textbook. (We will not be working with oblique prisms, which have parallelograms for the lateral faces.)



If you know the area of one of the bases, multiplying it by the height gives you the volume of the prism. In the formula below, we are using a capital B to represent the *area* of the base.

Volume of a Prism

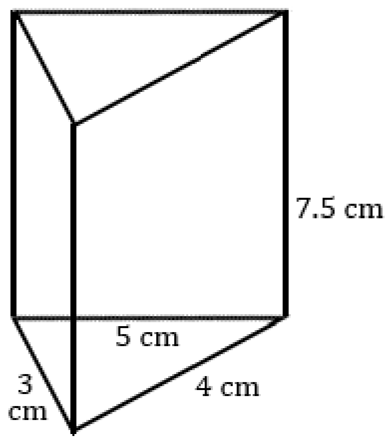
For a prism with base area B and height h:

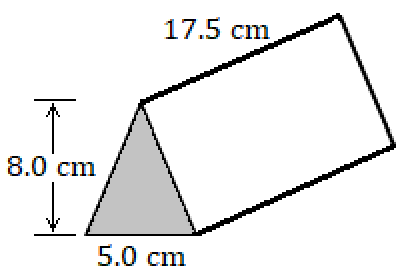
$V = Bh$

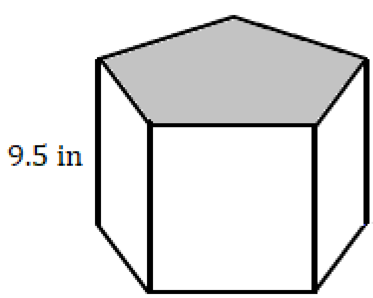
If the prism is lying on its side, the "height" will look like a length. No matter how the prism is oriented, the height is the dimension that is perpendicular to the planes of the two parallel bases.

Exercises

Find the volume of each prism.

3. 

4. 

5. 

The area of the pentagon is 55 square inches.

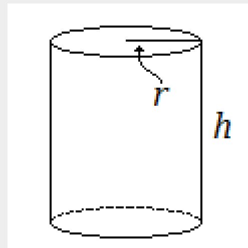
Volume: Cylinders

A cylinder can be thought of as a prism with bases that are circles, rather than polygons. Just as with a prism, the volume is the area of the base multiplied by the height.

Volume of a Cylinder

For a cylinder with radius r (or diameter d) and height h:

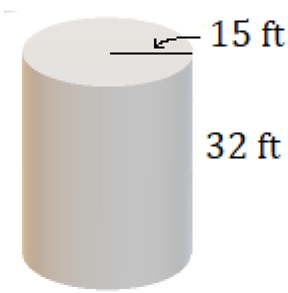
$V = \pi r^2 h$ or $V = \pi d^2 h \div 4$



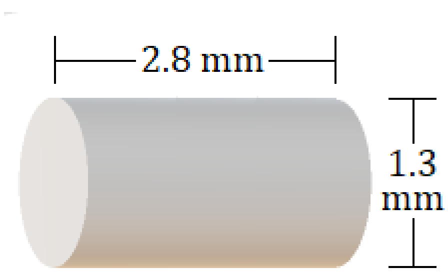
Exercises

Find the volume of each cylinder.

6.

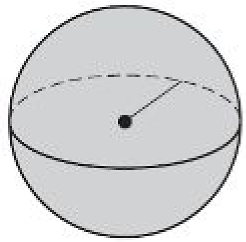


7.



Volume: Spheres

As with surface area, we would need to use calculus to derive the formula for the volume of a **sphere**. Just believe it. ¯_(ツ)_/¯



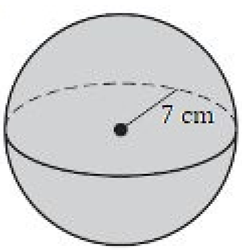
Volume of a Sphere

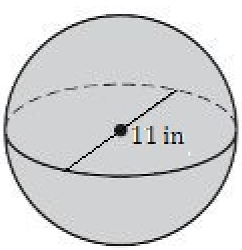
For a sphere with radius r or diameter d:

$V = 4\pi r^3 \div 3$ or $V = \pi d^3 \div 6$

Exercises

Find the volume of each sphere.

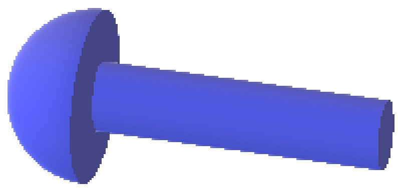
8. 

9. 

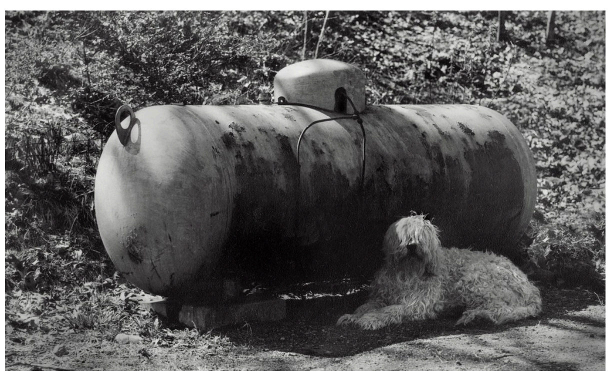
Volume: Composite Solids

Of course, not every three-dimensional object is a prism, cylinder, or sphere. A composite solid is made up of two or more simpler solids. As with two-dimensional composite figures, breaking the figure into recognizable solids is a good first step.

Exercises



10. A rivet is formed by topping a cylinder with a hemisphere. The width of the cylindrical part (the rivet pin) is 1.6 cm and the length is 7.0 cm. The width of the hemisphere-shaped top (the rivet head) is 3.2 cm. Find the rivet's volume.



Casper relaxes in the shadow of the propane tank.
Photo by David Dodge on flickr.

A 250-gallon propane tank is roughly in the shape of a cylinder with a hemisphere on each end. The length of the cylindrical part is 6.0 feet long, and the cross-sectional diameter of the tank is 2.5 feet.

11. Calculate the volume of the tank in cubic feet.

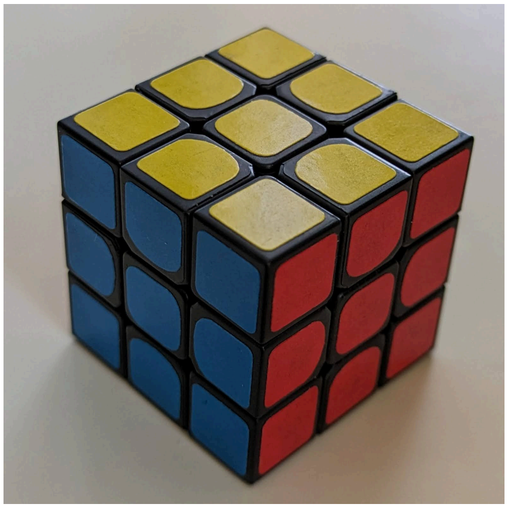
12. Verify that the tank can hold 250 gallons of liquid propane.

Exercise Answers

[25]

Converting Units of Volume

You may use a calculator throughout this module.



Just as we saw with area, converting between units of volume requires us to be careful because cubic units behave differently than linear units.

Quantities of mulch, dirt, or gravel are often measured by the cubic yard. How many cubic feet are in one cubic yard?

1 yard = 3 feet, so we can divide the length into three sections, the width into three sections, and the height into three sections to convert all three dimensions of the cube from yards to feet. This forms a 3 by 3 by 3 cube, which shows us that 1 cubic yard equals 27 cubic feet. The linear conversion ratio of 1 to 3 means that that the conversion ratio for the volumes is 1 to 3^3, or 1 to 27.

Here's another way to think about it without a diagram: $1 \text{ yd} = 3 \text{ ft}$, so $(1 \text{ yd})^3 = (3 \text{ ft})^3$. To remove the parentheses, we must cube the number *and* cube the units: $(3 \text{ ft})^3 = 3^3 \text{ ft}^3 = 27 \text{ ft}^3$.

More generally, we need to **cube** the linear conversion factors when converting units of volume. If the linear units have a ratio of 1 to n, the cubic units will have a ratio of 1 to n^3.

U.S. System: Converting Measurements of Volume

$$1 \text{ ft}^3 = 1,728 \text{ in}^3$$

$$1 \text{ yd}^3 = 27 \text{ ft}^3$$

$$1 \text{ yd}^3 = 46,656 \text{ in}^3$$

Exercises

1. True story: A friend at the National Guard base gave us <u>three</u> long wooden crates to use as raised planting beds. (The crates probably carried some kind of weapons or ammunition, but our friend wouldn't say.) Henry, who was taking geometry in high school, was asked to measure the crates and figure out how much soil we needed. The inside dimensions of each crate were 112 inches long, 14 inches wide, and 14 inches deep. We wanted to fill them most of the way full with soil, leaving about 4 inches empty at the top. How many cubic yards of soil did we need to order from the supplier?

2. True story, continued: We decided to check our answer and did a rough estimate by rounding each dimension to the nearest foot, then figuring out the volume from there. Did this give the same result?

This is one of the three crates.

We can convert between units of volume and liquid capacity. As you might expect, the numbers are messy in the U.S. system.

$$1 \text{ fl oz} \approx 1.8047 \text{ in}^3$$
$$1 \text{ ft}^3 \approx 7.4805 \text{ gal}$$

Exercises

3. A circular wading pool has a diameter of roughly 5 feet and a depth of 6 inches. How many gallons of water are required to fill it about 80% of the way full?

4. A standard U.S. soda pop can has a diameter of $2\frac{1}{2}$ inches and a height of $4\frac{3}{4}$ inches. Verify that the can is able to hold 12 fluid ounces of liquid.

5. A water jug that is roughly cube-shaped is advertised with dimensions 10.51 inches deep, 10.51 inches wide, and 10.51 inches high. How many gallons is its capacity?

Tiger the cat couldn't jump onto the storage container, so I made him a staircase. He didn't understand the concept.

Metric System: Converting Measurements of Volume

$$1 \text{ cm}^3 = 1 \text{ cc} = 1 \text{ mL}$$
$$1 \text{ cm}^3 = 1,000 \text{ mm}^3$$
$$1 \text{ m}^3 = 1,000,000 \text{ cm}^3$$
$$1 \text{ L} = 1,000 \text{ cm}^3$$
$$1 \text{ m}^3 = 1,000 \text{ L}$$

It's no surprise that the metric conversion ratios for volume are all powers of 10 ... but they are actually powers of $1,000$, or 10^3, because the linear conversions get cubed when we go three-dimensional.

Exercises

6. The circular soaking tubs at Hot Lake Springs in La Grande, Oregon have an inner diameter of 2.00 meters, and the water has a depth of 0.75 meters. What is the approximate volume of water, in liters, in one of the tubs?

7. A 1.5 L carton of orange juice has a square base 9.5 centimeters on each side, and the height of the rectangular section is 19 centimeters. (Ignore the triangular section at the top.) Verify that the carton holds 1.5 liters of juice.

8. A 1.89 L carton of POG juice has a square base 9.5 centimeters on each side, and the height of the rectangular section is 19.5 centimeters. This is not enough volume to hold 1.89 liters of juice unless the sides of the carton bulge out. If you unfold the triangular top section to make a rectangular column and squeeze the sides of the carton so they aren't bulging out, what height will the juice rise to?

Both Systems: Converting Measurements of Volume

Converting volumes between the U.S. and metric systems will of course involve messy decimal values. For example, because $1 \text{ in} = 2.54 \text{ cm}$, we can cube both numbers and find that $1 \text{ in}^3 = (2.54 \text{ cm})^3 = 16.387064 \text{ cm}^3$. The conversions in the table below are rounded to five significant figures.

$$1 \text{ in}^3 \approx 16.387 \text{ cm}^3$$
$$1 \text{ m}^3 \approx 35.315 \text{ ft}^3$$
$$1 \text{ m}^3 \approx 1.3080 \text{ yd}^3$$

Exercises

9. A large dumpster in Iceland has a volume of 15 cubic meters. Convert this to cubic yards.

10. Suppose you need to know the volume of an Icelandic dumpster in cubic feet. Convert $15\ \mathrm{m}^3$ to ft^3.

Hvaða rusl!

11. A "two yard" dumpster at a campground in Washington has a volume of 2 cubic yards. Convert this to cubic meters.

What rubbish!

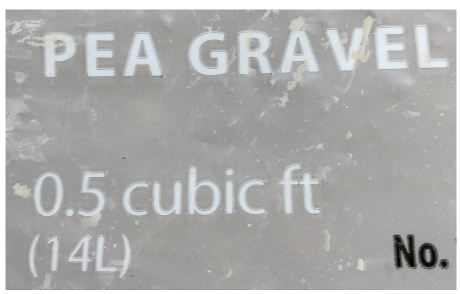
12. The engine of a 1964 Corvette has 8 cylinders with a bore (diameter) of 4.00 inches and a stroke (height) of 3.25 inches. Find the total displacement (volume) of the cylinders in this engine, rounded to the nearest cubic inch.[1]

13. Convert the displacement of the Corvette engine into cubic centimeters.

14. Convert the displacement of the Corvette engine into liters.

Photo by Chevrolet via Wikimedia Commons

15. A hardware store sells pea gravel in $0.5\ \text{ft}^3$ bags. The bags are also marked $14\ \text{L}$. Verify that these two measures are equivalent.



Density

The density of a material is its weight per volume such as pounds per cubic foot, or mass per volume such as grams per cubic centimeter. Multiplying the volume of an object by its density will give its weight or mass.

Exercises

16. A 0.5 ft^3 bag of pea gravel weighs approximately 50 pounds. Determine the density of pea gravel in pounds per cubic foot.

17. The standard size of a gold bar in the U.S. Federal Reserve is 7 inches by $3\frac{5}{8}$ inches by $1\frac{3}{4}$ inches.[2] The density of gold is 0.698 pounds per cubic inch. How much does one gold bar weigh?

18. A cylindrical iron bar has a diameter of 3.0 centimeters and a length of 20.0 centimeters. The density of iron is 7.87 grams per cubic centimeter. What is the bar's mass, in kilograms?

Volumes of Similar Solids

Earlier in this module, it was stated that if the linear units have a ratio of 1 to n, the cubic units will have a ratio of 1 to n^3. This applies to similar solids as well.

If the linear dimensions of two similar solids have a ratio of 1 to n, then the volumes will have a ratio of 1 to n^3.

We'll verify this in the following exercises.

Exercises

A table tennis (ping pong) ball has a diameter of 4 centimeters. A wiffle® ball has a diameter twice that of a table tennis ball.

19. Determine the volume of the wiffle® ball.

20. Determine the volume of the table tennis ball.

21. What is the ratio of the volumes of the two balls?

Rectangular solid A has dimensions 3 inches by 4 inches by 5 inches. Rectangular solid B has dimensions triple those of A's.

22. Determine the volume of the larger solid, B.

23. Determine the volume of the smaller solid, A.

24. What is the ratio of the volumes of the two solids?

Exercise Answers

Notes

1. The formula is often written as $V = \frac{\pi}{4} \cdot b^2 \cdot s \cdot c$, where c is the number of cylinders.
2. Source: https://www.usmint.gov/about/mint-tours-facilities/fort-knox

[26]

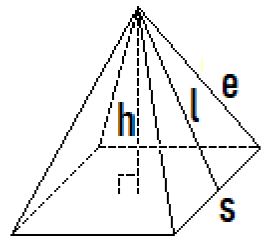
Pyramids and Cones

You may use a calculator throughout this module.

Pyramids

The pyramid complex at Giza, Egypt. Photo by Ricardo Liberato on Wikipedia

A **pyramid** is a geometric solid with a polygon base and triangular faces with a common vertex (called the *apex* of the pyramid). Pyramids are named according to the shape of their bases. The most common pyramids have a square or another regular polygon for a base, making all of the faces identical isosceles triangles. The height, h, is the distance from the apex straight down to the center of the base. Two other measures used with pyramids are the edge length e, the sides of the triangular faces, and the slant height l, the height of the triangular faces.



Volume of a Pyramid

In general, the volume of a pyramid with base of area B and height h is

$V = \frac{1}{3}Bh$ or $V = Bh \div 3$

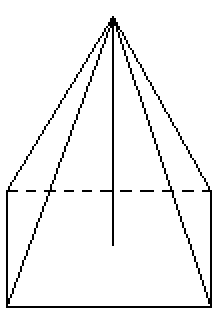
If the base is a square with side length s, the volume is

$V = \frac{1}{3}s^2h$ or $V = s^2h \div 3$

Interestingly, the volume of a pyramid is $\frac{1}{3}$ the volume of a prism with the same base and height.

Exercises

1. A pyramid has a square base with sides 16.0 centimeters long, and a height of 15.0 centimeters. Find the volume of the pyramid.



2. The Great Pyramid at Giza has a height of 137 meters and a square base with sides $23\overline{0}$ meters long.[1] Find the volume of the pyramid.

The lateral surface area (LSA) of a pyramid is found by adding the area of each triangular face.

Lateral Surface Area of a Pyramid

If the base of a pyramid is a regular polygon with n sides each of length s, and the slant height is l, then

$LSA = \frac{1}{2}nsl$ or $LSA = nsl \div 2$

If the base is a square, then

$LSA = 2sl$

The total surface area (TSA) is of course found by adding the area of the base B to the lateral surface area. If the base is a regular polygon, you will need to use the techniques we studied in Module 23.

Total Surface Area of a Pyramid

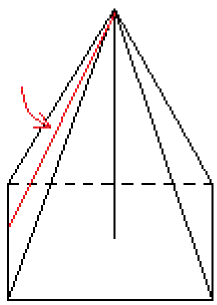
$TSA = LSA + B$

If the base is a square, then

$TSA = 2sl + s^2$

Exercises

3. A pyramid has a square base with sides 16.0 centimeters long, and a slant height of 17.0 centimeters. Find the lateral surface area and total surface area of the pyramid.



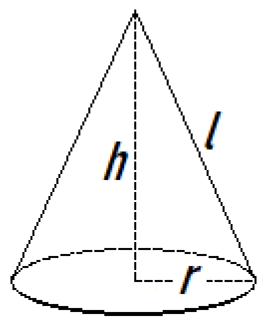
4. The Great Pyramid at Giza has a slant height of 179 meters and a square base with sides $23\overline{0}$ meters long. Find the lateral surface area of the pyramid.

Cones

A large cone-shaped pile of grain in Eastern Oregon.

A cone is like a pyramid with a circular base. You may be able to determine the height h of a cone (the altitude from the apex, perpendicular to the base), or the slant height l (which is the length from the apex to the edge of the circular base).

Note that the height, radius, and slant height form a right triangle with the slant height as the hypotenuse. We can use the Pythagorean theorem to determine the following equivalences.



The slant height l, height h, and radius r of a cone are related as follows:

$l = \sqrt{r^2 + h^2}$

$h = \sqrt{l^2 - r^2}$

$r = \sqrt{l^2 - h^2}$

Just as the volume of a pyramid is $\frac{1}{3}$ the volume of a prism with the same base and height, the volume of a cone is $\frac{1}{3}$ the volume of a cylinder with the same base and height.

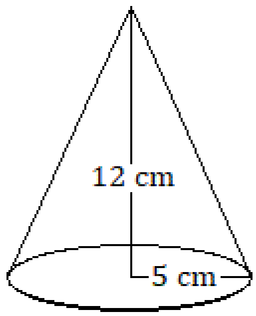
Volume of a Cone

The volume of a cone with a base radius r and height h is

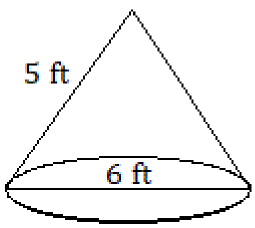
$V = \frac{1}{3}\pi r^2 h$ or $V = \pi r^2 h \div 3$

Exercises

5. The base of a cone has a radius of 5.0 centimeters, and the vertical height of the cone is 12.0 centimeters. Find the volume of the cone.



6. The base of a cone has a diameter of 6.0 feet, and the slant height of the cone is 5.0 feet. Find the volume of the cone.

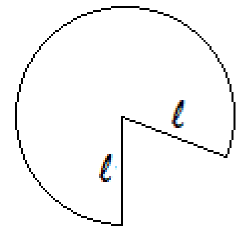


For the surface area of a cone, we have the following formulas.

Surface Area of a Cone

$$LSA = \pi r l$$

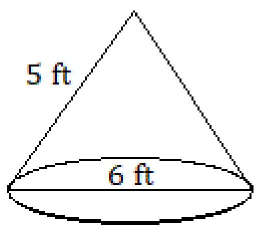
$$TSA = LSA + \pi r^2 = \pi r l + \pi r^2$$



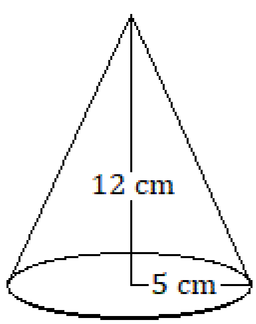
It's hard to explain the LSA formula in words, but here goes. The lateral surface of a cone, when flattened out, is a circle with radius l that is missing a wedge. The circumference of this partial circle, because it matched the circumference of the circular base, is $2\pi r$. The circumference of the entire circle with radius l would be $2\pi l$, so the part we have is just a fraction of the entire circle. To be precise, the fraction is $\frac{2\pi r}{2\pi l}$, which reduces to $\frac{r}{l}$. The area of the entire circle with radius l would be πl^2. Because the partial circle is the fraction $\frac{r}{l}$ of the entire circle, the area of the partial circle is $\pi l^2 \cdot \frac{r}{l} = \pi r l$.

Exercises

7. The base of a cone has a diameter of 6.0 feet, and the slant height of the cone is 5.0 feet. Find the lateral surface area and total surface area of the cone.



8. The base of a cone has a radius of 5.0 centimeters, and the vertical height of the cone is 12.0 centimeters. Find the lateral surface area and total surface area of the cone.



Exercise Answers

Notes

1. https://en.wikipedia.org/wiki/Great_Pyramid_of_Giza

[27]

Percents Part 3

Photo by Karim Manjra on Unsplash

You may use a calculator throughout this module.

There is one more situation involving percents that often trips people up: working backwards from the result of a percent change to find the original value.

$$\text{Amount} = \text{Rate} \cdot \text{Base}$$

$$A = R \cdot B$$

Finding the Base After a Percent Increase

Suppose a 12% tax is added to a price; what percent of the original is the new amount?

Well, the original number is 100% of itself, so the new amount must be $100\% + 12\% = 112\%$ of the original.

As a proportion, $\frac{A}{B} = \frac{112}{100}$.

As an equation, $A = 1.12 \cdot B$.

A very common error is to find 12% of the new amount and subtract that from the new amount. However, this doesn't give the correct result. Instead, we must ***divide*** the new amount by 112%.

If an unknown number is increased by a percent, add that percent to 100% and use that result for the rate.

$$\frac{\text{new amount}}{\text{unknown base}} = \frac{100+\text{percent}}{100}$$

Exercises

1. A sales tax of 8% is added to the selling price of a lawn tractor, making the total price $\$1,402.92$. What is the selling price of the lawn tractor without tax?
2. Clackamas Community College's enrollment in Fall 2023 was $17,605$ students, which was an increase of 18.465% from Fall 2022. What was the enrollment in Fall 2022?
3. While your author and his children were stuck overnight at the Seattle airport, they bought three 20-ounce bottles of Sprite for $\$11.86$, but the prices weren't shown and they weren't given a receipt. Suspecting that the sales tax rate was 10.1% from previous experience, your author was able to figure out the cost of one Sprite. Can you?

Finding the Base After a Percent Decrease

Suppose a 12% discount is applied to a price; what percent of the original is the new amount?

As above, the original number is 100% of itself, so the new amount must be $100\% - 12\% = 88\%$ of the original.

As a proportion, $\frac{A}{B} = \frac{88}{100}$.

As an equation, $A = 0.88 \cdot B$.

As above, the most common error people make is finding 12% of the new amount and adding that to the new amount, but this doesn't give the correct result. Instead, we must ***divide*** the new amount by 88%.

If an unknown number is **decreased** by a percent, subtract that percent from 100% and use that result for the rate.

$$\frac{\text{new amount}}{\text{unknown base}} = \frac{100-\text{percent}}{100}$$

Exercises

4. A city department's budget was cut by 5.00% this year. If this year's budget is $\$3.04$ million, what was last year's budget?

5. The estimated population of San Francisco in July 2022 was $808,400$ people, which was a decrease of 7.50% from April 2020. What was the estimated population in April 2020? (Round to the nearest hundred people.)[1]

6. An educational website claims that by purchasing access for $\$5$, you'll save 69% off the standard price. What is the standard price?

Exercise Answers

Notes

1. Source: https://www.census.gov/quickfacts/fact/table/sanfranciscocitycalifornia/PST045222

[28]

Mean, Median, Mode

You may use a calculator throughout this module.

We often describe data using a ***measure of central tendency***. This is a number that we use to describe the typical data value. In this module, we will look at three measures of central tendency: the ***mean***, the ***median***, and the ***mode***. Each of these has pros and cons, depending on the particular data set.

Mean

The mean of a set of data is what we commonly call the average: add up all of the numbers and then divide by how many numbers there were.

Exercises

1. The table below shows the amount of time, rounded to the nearest half minute, it took Marty to complete the Friday crossword puzzle in the New York Times. Calculate the mean completion time for these thirteen puzzles.

Oct 6, 2023	10.0 min
Oct 13, 2023	13.0 min
Oct 20, 2023	11.0 min
Oct 27, 2023	9.0 min
Nov 3, 2023	8.5 min
Nov 10, 2023	9.5 min
Nov 17, 2023	11.0 min
Nov 24, 2023	12.0 min
Dec 1, 2023	11.5 min
Dec 8, 2023	9.5 min
Dec 15, 2023	11.0 min
Dec 22, 2023	11.0 min
Dec 29, 2023	7.0 min

2. The table below shows the average price of a gallon of regular unleaded gasoline in the Seattle metro area for ten weeks in late 2023.[1] Compute the mean price over this time period.

Oct 23, 2023	\$4.81
Oct 30, 2023	\$4.70
Nov 6, 2023	\$4.63
Nov 13, 2023	\$4.57
Nov 20, 2023	\$4.49
Nov 27, 2023	\$4.45
Dec 4, 2023	\$4.39
Dec 11, 2023	\$4.34
Dec 18, 2023	\$4.28
Dec 25, 2023	\$4.22

Median

The median is the middle number in a set of data; it has an equal number of data values below it as above it. The numbers must be arranged in order, usually smallest to largest but largest to smallest would also work. Then we can count in from both ends of the list and find the median in the middle.

If there are an odd number of data values, there will be one number in the middle, which is the median.

If there are an even number of data values, there will be two numbers in the middle. The mean of these two numbers is the median.

Exercises

3. Here are Marty's Friday crossword puzzle completion times again, in minutes, listed in order from fastest to slowest. What is the median completion time? $7.0, 8.5, 9.0, 9.5, 9.5, 10.0, 11.0, 11.0, 11.0, 11.0, 11.5, 12.0, 13.0$

4. Here are the Seattle gas prices again, listed in order from lowest to highest. What is the median price?$\$4.22$, $\$4.28$, $\$4.34$, $\$4.39$, $\$4.45$, $\$4.49$, $\$4.57$, $\$4.63$, $\$4.70$, $\$4.81$

The five houses on a block have these property values: $\$250,000$; $\$300,000$; $\$320,000$; $\$190,000$; $\$220,000$.

5. Find the mean property value.

6. Find the median property value.

A new house is built on the block, making the property values $\$250,000$; $\$300,000$; $\$320,000$; $\$190,000$; $\$220,000$; $\$750,000$.

7. Find the mean property value.

8. Find the median property value.

9. Which of these measures appears to give a more accurate representation of the typical house on the block?

The mean is better to work with when we do more complicated statistical analysis, but it is sensitive to extreme values; in other words, one very large or very small number can have a significant effect on the mean. The median is not sensitive to extreme values, which can make it a better measure to use when describing data that has one or two numbers very different from the remainder of the data.

Mode

The mode is the value that appears **most** frequently in the data set. On the game show *Family Feud*, the goal is to guess the mode: the most popular answer.

If no numbers are repeated, then the data set has no mode. If there are two values that are tied for most frequently occurring, then they are both considered a mode and the data set is called ***bimodal***. If there are more than two values tied for the lead, we usually say that there is no mode.[2] (It's like in sports: there is usually one MVP, but occasionally there are two co-MVPs. Having three or more MVPs would start to get ridiculous.)

Exercises

10. Here are Marty's Friday crossword puzzle completion times one last time, in minutes. What is the mode of the completion times? 7.0, 8.5, 9.0, 9.5, 9.5, 10.0, 11.0, 11.0, 11.0, 11.0, 11.5, 12.0, 13.0

11. Here are the Seattle gas prices one last time. What is the mode of the prices? \$4.22, \$4.28, \$4.34, \$4.39, \$4.45, \$4.49, \$4.57, \$4.63, \$4.70, \$4.81

12. One hundred cell phone owners are asked which carrier they use. What is the mode of the data?

AT&T Mobility	Verizon Wireless	T-Mobile US	Dish Wireless	U.S. Cellular
43	29	24	2	2

13. Fifty people are asked what their favorite type of Girl Scout cookie is. What is the mode?

S'Mores	Samoas	Tagalongs	Trefoils	Thin Mints
4	16	5	9	16

Let's put it all together and find the mean, median, and mode of some data sets. Sportsball!

Exercises

From 2001-2019, these are the numbers of games won by the New England Patriots each NFL season.[3]
$11, 9, 14, 14, 10, 12, 16, 11, 10, 14, 13, 12, 12, 12, 12, 14, 13, 11, 12.$

14. Find the mean number of games won from 2001 to 2019.

15. Find the median number of games won from 2001 to 2019.

16. Find the mode of the number of games won from 2001 to 2019.

17. Do any of these measures appear to be misleading, or do they all represent the data fairly well?

From 2001-2019, these are the numbers of games won by the Buffalo Bills each NFL season.[4]
$3, 8, 6, 9, 5, 7, 7, 7, 6, 4, 6, 6, 6, 9, 8, 7, 9, 6, 10.$

18. Find the mean number of games won from 2001 to 2019.

19. Find the median number of games won from 2001 to 2019.

20. Find the mode of the number of games won from 2001 to 2019.

21. Do any of these measures appear to be misleading, or do they all represent the data fairly well?

Some sets of data may not be easy to describe with one measure of central tendency.

Exercises

Thirteen clementines are weighed. Their masses, in grams, are
$82, 90, 90, 92, 93, 94, 94, 102, 107, 107, 108, 109, 109.$

22. Determine the mean. Does the mean appear to represent the mass of a typical clementine?

23. Determine the median. Does the median appear to represent the mass of a typical clementine?

24. Determine the mode. Does the mode appear to represent the mass of a typical clementine?

Suppose that the 108-gram clementine is a tiny bit heavier and the masses are actually
$82, 90, 90, 92, 93, 94, 94, 102, 107, 107, 109, 109, 109.$

25. Determine the new mean. Is the new mean different from the original mean?

26. Determine the new median. Is the new median different from the original median?

27. Determine the new mode. Is the new mode different from the original mode? Does it represent the mass of a typical clementine?

Clementine the cat weighs more than 109 grams.

Exercise Answers

Notes

1. Source: https://www.eia.gov/petroleum/gasdiesel/
2. The concepts of trimodal and multimodal data exist, but we aren't going to consider anything beyond bimodal in this textbook.
3. Source: https://www.pro-football-reference.com/teams/nwe/index.htm
4. Source: https://www.pro-football-reference.com/teams/buf/index.htm

[29]

Probability

Probability is the likelihood that some event occurs. If the event occurs, we call that a ***favorable outcome***. The set of all possible events (or outcomes) is called the ***sample space*** of the event.

We will limit our focus to ***independent*** events, which do not influence each other. For example, if we roll a 5 on one die, that does not affect the probability of rolling a 5 on the other die. (We will not be studying *dependent* events, which do influence each other.)

Theoretical Probability

If we are working with something simple like dice, cards, or coin flips where we know all of the possible outcomes, we can calculate the ***theoretical probability*** of an event occurring. To do this, we divide the number of ways the event can occur by the total number of possible outcomes. We may choose to write the probability as a fraction, a decimal, or a percent depending on what form seems most useful.

Theoretical probability of an event:

$$P(\text{event}) = \frac{\text{number of favorable outcomes}}{\text{total number of outcomes}}$$

Suppose two six-sided dice, numbered 1 through 6, are rolled. There are $6 \cdot 6 = 36$ possible outcomes in the sample space. If we are playing a game where we take the sum of the dice, the only possible outcomes are 2 through 12. However, as the following table shows, these outcomes are not all equally likely. For example, there are two different ways to roll a 3, but only one way to roll a 2.

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Exercises

Two six-sided dice numbered 1 through 6 are rolled. Find the probability of each event occurring.

1. The sum of the dice is 7.
2. The sum of the dice is 11.
3. The sum of the dice is 7 or 11.
4. The sum of the dice is greater than 1.
5. The sum of the dice is 13.

Some things to notice...

If an event is impossible, its probability is 0% or 0.

If an event is certain to happen, its probability is 100% or 1.

If it will be tedious to count up all of the favorable outcomes, it may be easier to count up the unfavorable outcomes and subtract from the total.

Exercises

Two six-sided dice numbered 1 through 6 are rolled. Find the probability of each event occurring.

6. The sum of the dice is 5.

7. The sum of the dice is not 5.

8. The sum of the dice is greater than 9.

9. The sum of the dice is 9 or lower.

The set of outcomes in which an event does not occur is called the ***complement*** of the event. The event "the sum is not *5*" is the complement of "the sum is *5*". Two complements *complete* the sample space.

If the probability of an event happening is p, the probability of the complement is $1 - p$.

Exercises

A bowl of 60 Tootsie Rolls Fruit Chews contains the following: 15 cherry, 14 lemon, 13 lime, 11 orange, 7 vanilla.

10. If one Tootsie Roll is randomly selected from the bowl, what is the probability that it is cherry?

11. If one Tootsie Roll is randomly selected from the bowl, what is the probability that it is not cherry?

12. What is the probability that a randomly selected Tootsie Roll is either orange or vanilla?

13. What is the probability that a randomly selected Tootsie Roll is not orange or vanilla?

Here's where we try to condense the basics of genetic crosses into one paragraph.

Each parent gives one allele to their child. The allele for brown eyes is B, and the allele for blue eyes is b. If two parents both have genotype Bb, the table below (which biologists call a Punnett square) shows that there are four equally-likely outcomes: BB, Bb, Bb, bb. The allele for brown eyes, B, is dominant over the gene for blue eyes, b, which means that if a child has any B alleles, they will have brown eyes. The only genotype for which the child will have blue eyes is bb.

	B	b
B	BB	Bb
b	Bb	bb

Exercises

Two parents have genotypes Bb and Bb. (B = brown, b = blue)

14. What is the probability that their child will have blue eyes?

15. What is the probability that their child will have brown eyes?

Now suppose that one parent has genotype Bb but the other parent has genotype bb. The Punnett square will look like this.

	B	b
b	Bb	bb
b	Bb	bb

Exercises

Two parents have genotypes *Bb* and *bb*. (*B* = brown, *b* = blue)

16. What is the probability that their child will have blue eyes?

17. What is the probability that their child will have brown eyes?

Empirical Probability

The previous methods work when we know the total number of outcomes and we can assume that they are all equally likely. (The dice aren't loaded, for example.) However, life is usually more complicated than a game of dice or a bowl of Tootsie Rolls. In many situations, we have to observe what has happened in the past and use that data to predict what might happen in the future. If someone predicts that an Alaska Airlines flight has an 85.4% of arriving on time[1], that is of course based on Alaska's past rate of success and this number will vary from month to month. When we calculate the probability this way, by observation, we call it an ***empirical probability***.

Empirical probability of an event:

$$P(\text{event}) = \frac{\text{number of favorable observations}}{\text{total number of observations}}$$

Although the wording may seem complicated, we are still just thinking about $\frac{\text{part}}{\text{whole}}$.

Exercises

A photocopier makes 250 copies, but 8 of them are unacceptable because they have toner smeared on them.

18. What is the empirical probability that a copy will be unacceptable?

19. What is the empirical probability that a copy will be acceptable?

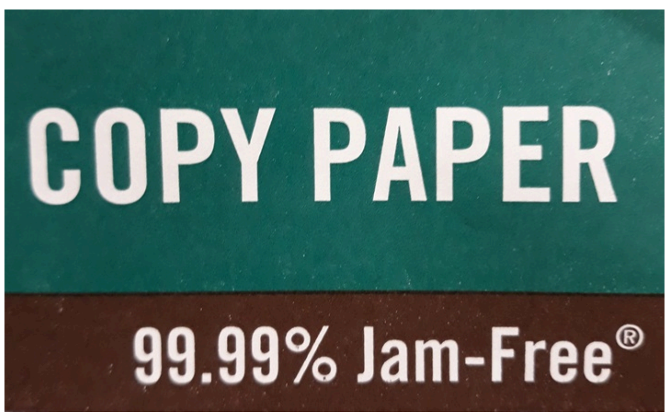
20. Out of the next $1,000$ copies, how many should we expect to be acceptable?

An auditor examined 200 tax returns and found errors on 44 of them.

21. What percent of the tax returns contained errors?

22. How many of the next $1,000$ tax returns should we expect to contain errors?

23. What is the probability that a given tax return, chosen at random, will contain errors?



Probability of More Than One Event

It was mentioned earlier in this module that ***independent events*** have no influence on each other. Some examples:

- Rolling two dice are independent events because the result of the first die does not affect the probability of what will happen with the second die.

- If we flip a coin ten times, each flip is independent of the previous flip because the coin doesn't remember how it landed before. The probability of heads or tails remains $\frac{1}{2}$ for each flip.
- Drawing marbles out of a bag are independent events only if we put the first marble back in the bag before drawing a second marble. If we draw two marbles at once, or we draw a second marble without replacing the first marble, these are *dependent events*, which we are not studying in this course.
- Drawing two cards from a deck of 52 cards are independent events only if we put the first card back in the deck before drawing a second card. If we draw a second card without replacing the first card, these are *dependent events*; the probabilities change because there are only 51 cards available on the second draw.

If two events are independent, then the probability of both events happening can be found by multiplying the probability of each event happening separately.

If A and B are independent events, then $P(A \text{ and } B) = P(A) \cdot P(B)$.

Note: This can be extended to three or more events. Just multiply all of the probabilities together.

Exercises

An auditor examined 200 tax returns and found errors on 44 of them.

24. What is the probability that the next two tax returns both contain errors?

25. What is the probability that the next three tax returns all contain errors?

26. What is the probability that the next tax return contains errors but the one after it does not?

27. What is the probability that the next tax return does not contain errors but the one after it does?

28. What is the probability that neither of the next two tax returns contain errors?
29. What is the probability that none of the next three tax returns contain errors?
30. What is the probability that at least one of the next three tax returns contain errors? (This one is tricky!)

Exercise Answers

Notes

1. Source (PDF file, see page 6): https://www.transportation.gov/sites/dot.gov/files/2024-01/December%202023%20ATCR.pdf

[30]

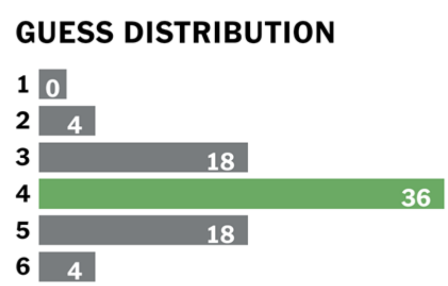
Standard Deviation

You probably won't need a calculator for this module.

This topic requires a leap of faith. It is one of the rare times when this textbook will say "don't worry about *why* it's true; just accept it." 😂

Normal Distributions & Standard Deviation

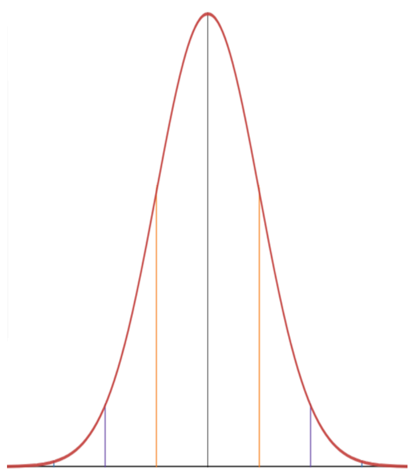
A ***normal distribution***, often referred to as a ***bell curve***, is symmetrical on the left and right, with the mean, median, and mode being the value in the center. There are lots of data values near the center, then fewer and fewer as the values get further from the center. A normal distribution describes the data in many real-world situations.



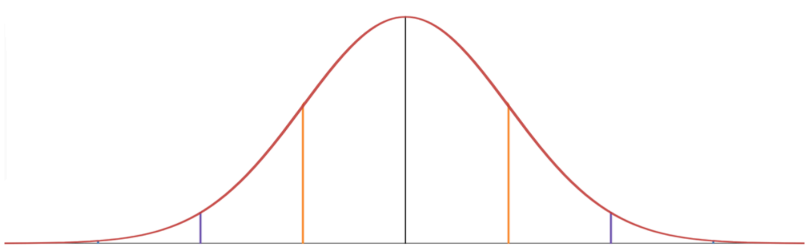
This normal distribution of Wordle scores is formatted horizontally.

One of the best ways to demonstrate the normal distribution is to drop balls through a board of evenly spaced pegs, as shown here. (The Plinko game on *The Price Is Right* is a well known example of this.) Each time a ball hits a peg, it has a fifty-fifty chance of going left or right. For most balls, the number of lefts and rights are roughly equal, and the ball lands near the center. Only a few balls have an extremely lop-sided number of lefts and rights, so there are not many balls at either end. As you can see, the distribution is not perfect, but it is approximated by the normal curve drawn on the glass.

The ***standard deviation*** is a measure of the spread of the data: data with lots of results close to the mean has a smaller standard deviation, and data with results spaced further from the mean has a larger standard deviation. (In this textbook, you will be given the value of the standard deviation of the data and will never need to calculate it.) The standard deviation is a measuring stick for a particular set of data.

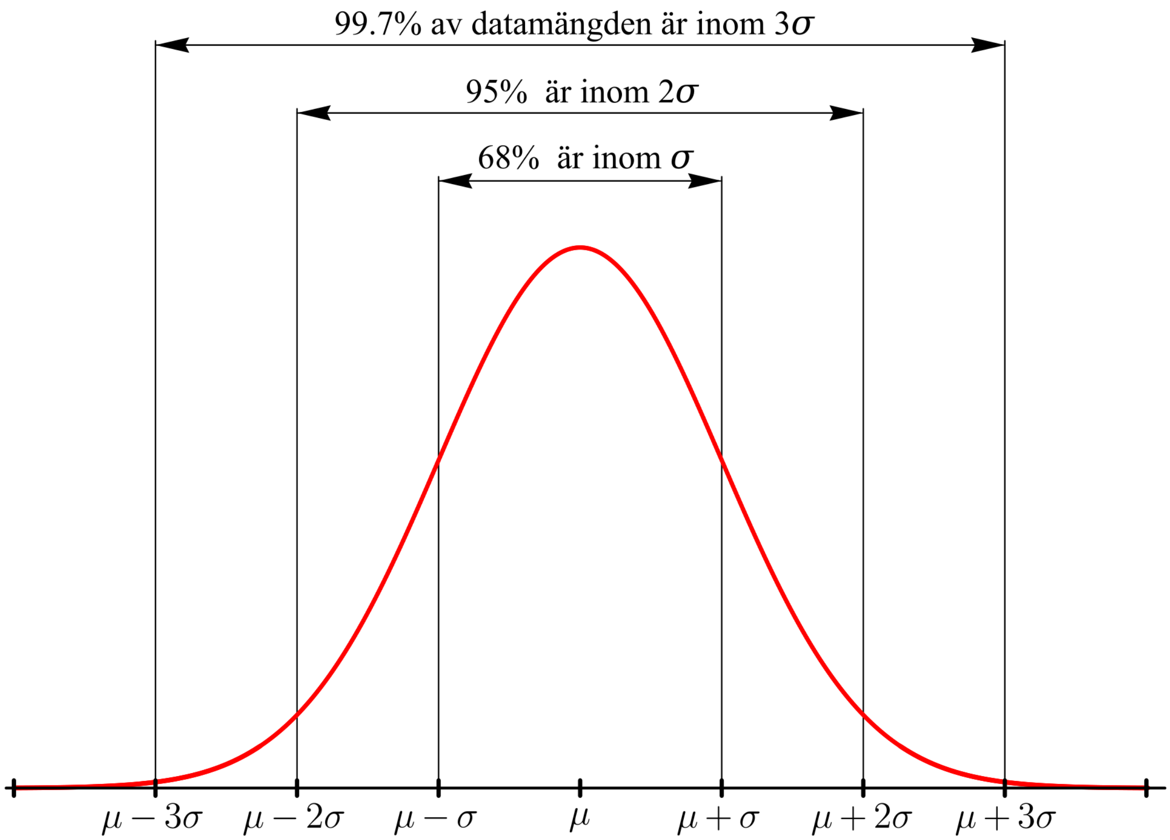


a distribution with a small standard deviation



a distribution with a large standard deviation

The 68-95-99.7 Rule



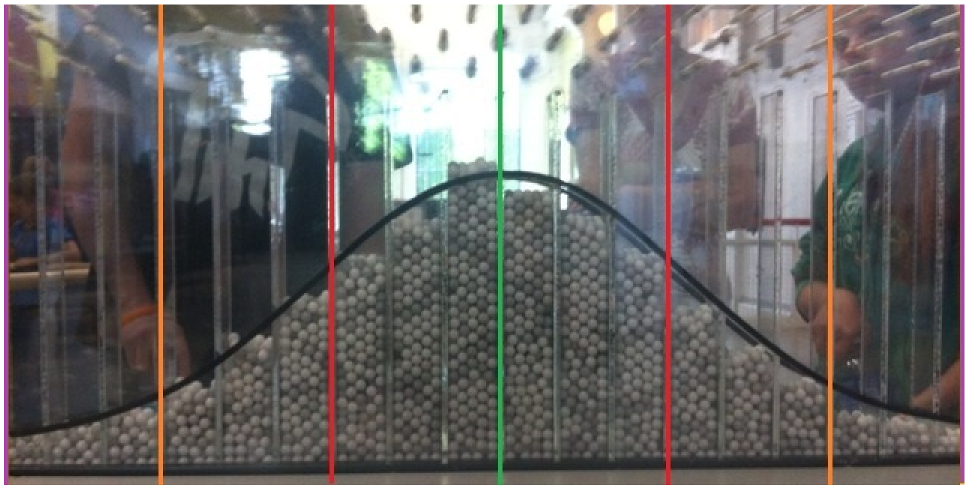
The 68-95-99.7 rule, in Swedish. Image credit: Svjo, hosted at Wikimedia Commons.

The 68-95-99.7 rule: In a normal distribution, approximately...

- 68% of the numbers are within 1 standard deviation above or below the mean
- 95% of the numbers are within 2 standard deviations above or below the mean
- 99.7% of the numbers are within 3 standard deviations above or below the mean

This is an *empirical* rule because it is based on observation of how the world works, rather than being based on a formula.[1]

Returning to the ball-dropping experiment, let's assume that the standard deviation is three columns wide.[2] In the picture below, the **green** line marks the center of the distribution.



First, the two **red** lines are each three columns away from the center, which is **one standard deviation** above and below the **center**, so about **68%** of the balls will land between the red lines.

Next, the two **orange** lines are another three columns farther away from the center, which is six columns or **two standard deviations** above and below the **center**, so about **95%** of the balls will land between the orange lines.

And finally, the two **purple** lines are another three columns farther away from the center, which is nine columns or **three standard deviations** above and below the **center**, so about **99.7%** of the balls will land between the purple lines. We can expect that 997 out of $1,000$ balls will land between the purple lines, leaving only 3 out of $1,000$ landing beyond the purple lines on either end.

Okay, that was a lot of information. For our purposes, the following restatement of the 68-95-99.7 rule may be more practical.

The 68-95-99.7 rule: In a normal distribution with mean μ and standard deviation σ...

- 68% of the numbers are between $\mu - \sigma$ and $\mu + \sigma$
- 95% of the numbers are between $\mu - 2\sigma$ and $\mu + 2\sigma$
- 99.7% of the numbers are between $\mu - 3\sigma$ and $\mu + 3\sigma$

Exercises

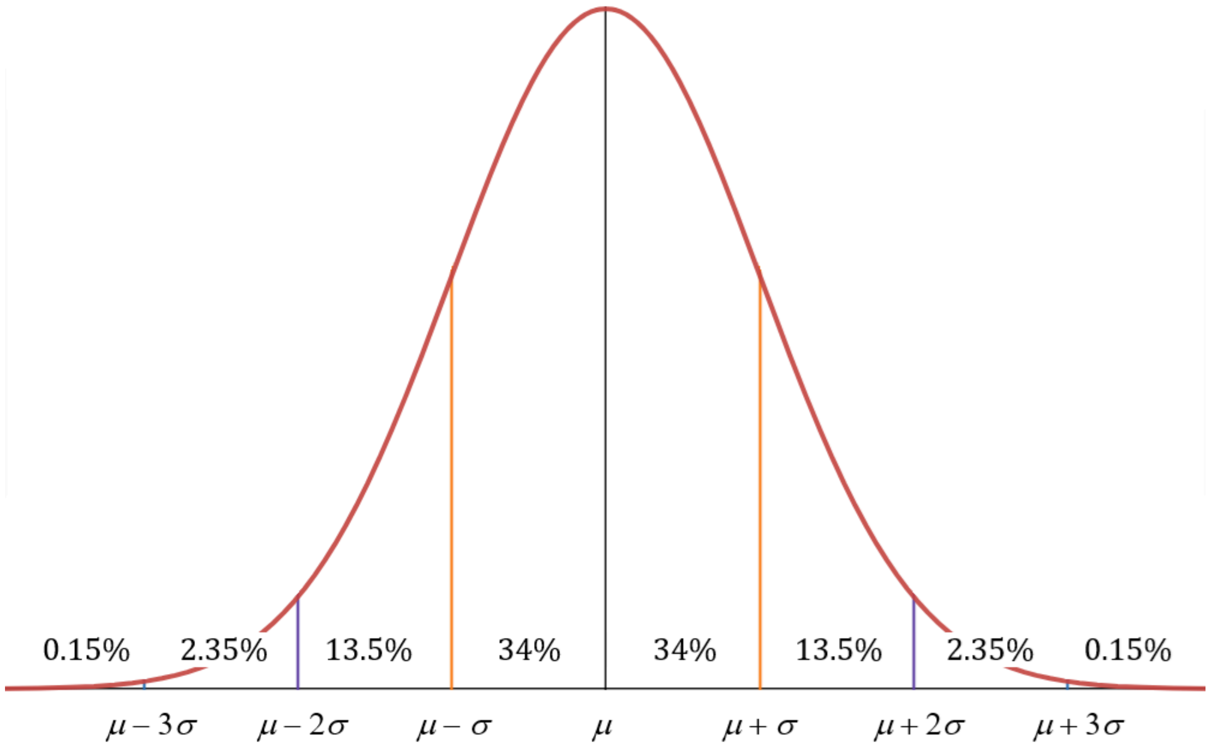
The heights of U.S. females are normally distributed. The average height is around 63.5 inches (5 ft 3.5 in) and the standard deviation is 3 inches. Use the 68-95-99.7 rule to fill in the blanks.

1. About 68% of the women should be between _______ and _______ inches tall.
2. About 95% of the women should be between _______ and _______ inches tall.
3. About 99.7% of the women should be between _______ and _______ inches tall.

The heights of U.S. males are normally distributed. The average height is around 69.5 inches (5 ft 9.5 in) and the standard deviation is 3 inches. Use the 68-95-99.7 rule to fill in the blanks.

4. About 68% of the men should be between _______ and _______ inches tall.
5. About 95% of the men should be between _______ and _______ inches tall.
6. About 99.7% of the men should be between _______ and _______ inches tall.

This graph provides another way to think about the distribution of the data.



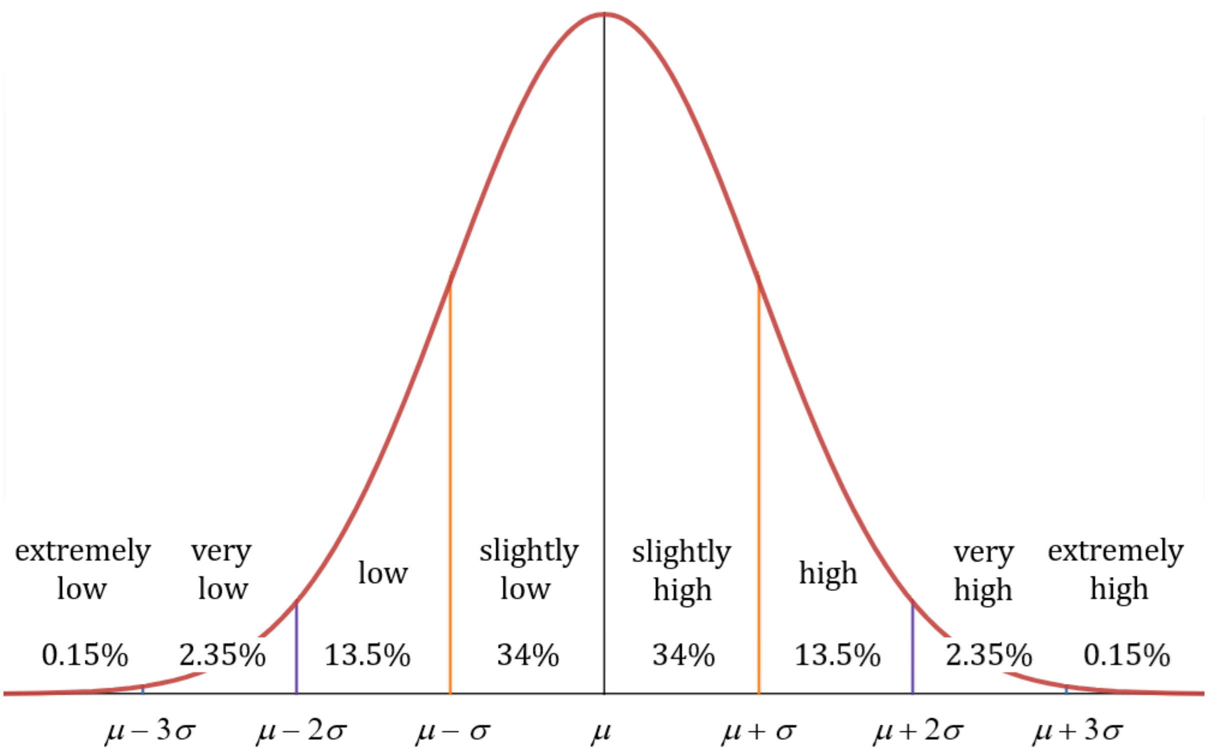
Because 68% of the data are within one standard deviation of the mean, we have 34% of the data slightly below the mean and 34% slightly above. Moving outwards one more standard deviation in each direction, we have another 13.5% below the mean and another 13.5% above the mean, encompassing a total of 95% of the data. Moving outwards one more standard deviation, we have another 2.35% far below the mean and another 2.35% far above the mean, bringing the total up to 99.7% of the data. This leaves only 0.15% of the data more than three standard deviations below the mean and 0.15% of the data more than three standard deviations above the mean.

Exercises

Around 16% of U.S. males in their forties weigh less than 160 lb and 16% weigh more than 230 lb. Assume a normal distribution.[3]

7. What percent of U.S. males weigh between 160 lb and 230 lb?
8. What is the average weight? (Hint: think about symmetry.)
9. What is the standard deviation? (Hint: You have to work backwards to figure this out, but the math isn't complicated.)
10. Based on the empirical rule, about 95% of the men should weigh between _______ and _______ pounds.

This version of the graph can help us group the data into general categories. This is not official terminology, but hopefully it gets the point across.



Looking at the middle 68% of the data, 34% could be considered "slightly low" and 34% could be considered "slightly high". Moving outwards, we have another

13.5% that could be considered "low" and another 13.5% could be considered "high". Moving outwards again, we have another 2.35% that could be considered "very low" and another 2.35% that could be considered "very high". Finally, 0.15% of the data could be considered "extremely low" and 0.15% could be considered "extremely high".

If you are asked only one question about the empirical rule instead of three in a row (68%, 95%, 99.7%), you will most likely be asked about the 95%. This is related to the "95% confidence interval" that is often mentioned in relation to statistics. For example, the margin of error for a poll is usually close to two standard deviations.[4]

Let's finish up by comparing the performance of three NFL teams at the beginning of this century.

Exercises

The numbers of regular-season games won by the New England Patriots[5] each NFL season from 2001-2019: 11, 9, 14, 14, 10, 12, 16, 11, 10, 14, 13, 12, 12, 12, 12, 14, 13, 11, 12.

The mean number of wins is 12.2, and a spreadsheet tells us that the standard deviation is 1.7 wins.

11. There is a 95% chance of the Patriots winning between _______ and _______ games in a season.

12. In 2020, the Patriots won 7 games. Could you have predicted that based on the data? How many standard deviations from the mean is this number of wins?

The numbers of regular-season games won by the Buffalo Bills[6] each NFL season from 2001-2019: 3, 8, 6, 9, 5, 7, 7, 7, 6, 4, 6, 6, 6, 9, 8, 7, 9, 6, 10.

The mean number of wins is 6.8, and a spreadsheet tells us that the standard deviation is 1.7 wins.

13. There is a 95% chance of the Bills winning between _______ and _______ games in a season.

14. In 2020, the Bills won 13 games. Could you have predicted that based on the data? How many standard deviations from the mean is this number of wins?

The numbers of regular-season games won by the Denver Broncos[7] each NFL season from 2001-2019: $8, 9, 10, 10, 13, 9, 7, 8, 8, 4, 8, 13, 13, 12, 12, 9, 5, 6, 7$.

The mean number of wins is 9.1, and a spreadsheet tells us that the standard deviation is 2.6 wins.

15. There is a 95% chance of the Broncos winning between _______ and _______ games in a season.

16. In 2020, the Broncos won 5 games. Could you have predicted that based on the data? How many standard deviations from the mean is this number of wins?

Exercise Answers

Notes

1. Well, there is a formula, $y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$, but it was discovered after the fact.
2. I eyeballed it and it seemed like a reasonable assumption.
3. Source (PDF file): https://www2.census.gov/library/publications/2010/compendia/statab/130ed/tables/11s0205.pdf
4. Source: https://en.wikipedia.org/wiki/Standard_deviation
5. Source: https://www.pro-football-reference.com/teams/nwe/index.htm
6. Source: https://www.pro-football-reference.com/teams/buf/index.htm
7. Source: https://www.pro-football-reference.com/teams/den/index.htm

[31]

Right Triangle Trigonometry

Bob Brown and Morgan Chase

You may use a calculator throughout this module.

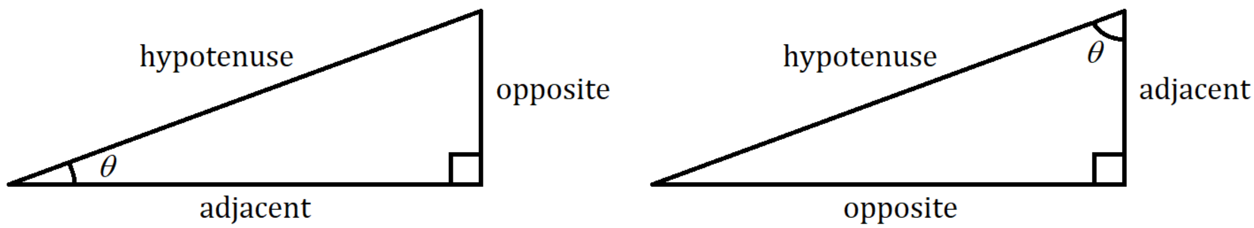
The word “trigonometry” essentially means the measurement of triangles. In this module, we will focus on right triangles. If we know some information about a right triangle, such as the measure of one side and one angle, we can use trigonometry to determine the measure of another side.

Photo by Georgios Karamanis on flickr.

Sine, Cosine, Tangent

Consider one of the acute angles in a right triangle.

- The *sine* of the angle is the ratio of the opposite side to the hypotenuse.
- The *cosine* of the angle is the ratio of the adjacent side to the hypotenuse.
- The *tangent* of the angle is the ratio of the opposite side to the adjacent side.



SOHCAHTOA

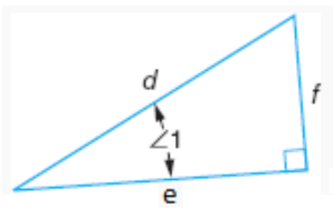
$$\text{sine} = \frac{\text{opposite}}{\text{hypotenuse}}$$

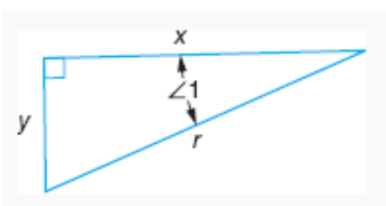
$$\text{cosine} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\text{tangent} = \frac{\text{opposite}}{\text{adjacent}}$$

Exercises

In relation to $\angle 1$, identify which side of the triangle is the adjacent side, the opposite side, and the hypotenuse.

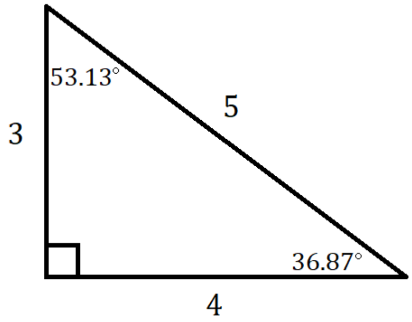
1. 

2. 

Now let's look at a specific example, a right triangle with sides of length 3, 4, and 5. Measuring the angles with a protractor will show us that the smaller acute angle is approximately $37°$ and the larger acute angle is $53°$; to the nearest hundredth of a degree, these values are $36.87°$ and $53.13°$.

Exercises

Use the diagram to determine each of the following, first as a fraction and then as a decimal (rounded to four significant figures, if necessary).



3. $\sin 36.87°$

4. $\cos 36.87°$

5. $\tan 36.87°$

6. $\sin 53.13°$

7. $\cos 53.13°$

8. $\tan 53.13°$

Note that we would have gotten the same results for any right triangle whose sides are in the ratio 3:4:5. If the sides had been 6, 8, 10, the angles measures would still be $36.87°$ and $53.13°$. If the sides had been 30, 40, 50, the angles measures would still be $36.87°$ and $53.13°$.

Or to look at it another way, any right triangle with angles $36.87°$ and $53.13°$ will have these relationships: the shorter leg is $\frac{3}{5}$ the hypotenuse, the longer leg is $\frac{4}{5}$ the hypotenuse, and the shorter leg is $\frac{3}{4}$ the longer leg. Your scientific calculator knows this and is programmed with all the trigonometric values you need.

Exercises

Use your calculator to determine each of the following as a decimal (rounded to four significant figures, if necessary). Be sure that your calculator is in degree mode, not radian mode, before you begin.

9. $\sin 36.87°$

10. $\cos 36.87°$

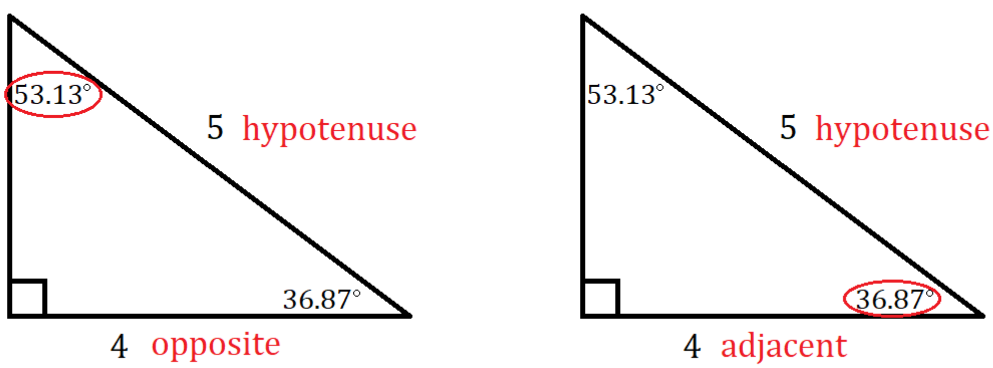
11. $\tan 36.87°$

12. $\sin 53.13°$

13. $\cos 53.13°$

14. $\tan 53.13°$

You may notice some repeated answers here, which makes sense because, for example, the side opposite the $53.13°$ angle is adjacent to the $36.87°$ angle, which means that $\sin 53.13°$ must be equal to $\cos 36.87°$.



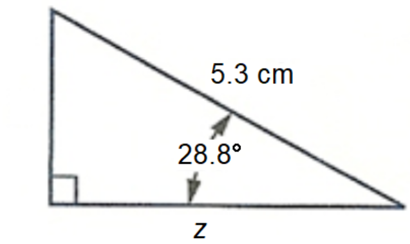
Trigonometry Applications

The sine, cosine, and tangent ratios apply to any right triangle, not just a 3-4-5 triangle, which is what makes trigonometry so useful. If we know the measure of one side and one acute angle in a right triangle, we can use SOHCAHTOA to find the length of another side of the triangle. For each of the following exercises, we'll need to decide whether the parts we're dealing with will require us to use sine, cosine, or tangent.

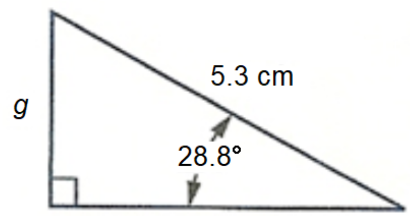
Exercises

Determine the length of the labeled unknown side of each right triangle. Round each answer appropriately.

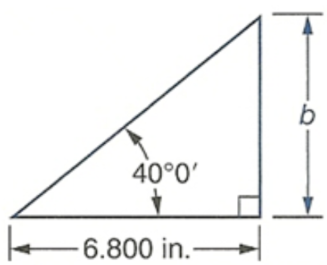
15.



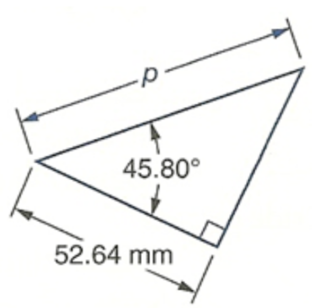
16.

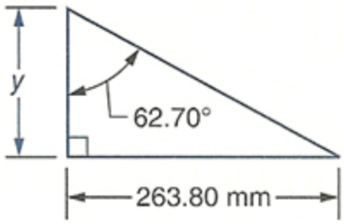


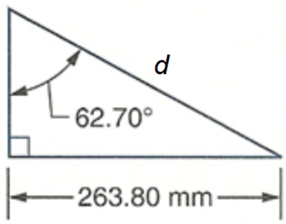
17.



18.



19. 

20. 

21. A guy wire is attached to a utility pole. The wire is anchored to the ground 15 feet from the base of the pole, forming a $56°$ angle with the level ground. What is the length of the wire, to the nearest foot?

22. A 19-foot ladder is leaning against a wall, forming a $70°$ angle with the level ground. What is the vertical height of the place where the ladder contacts the wall?

23. During intermission of a hockey game, one lucky fan gets to shoot a puck from the blue line at the goal, 64 feet away, for the chance to win a team jersey. Actually, there is a sheet of fabric blocking the entire goal except for a 6-inch wide hole in the bottom center. If the fan shoots the puck hard enough, aiming directly at the center of the hole, they win. If their aim is off by $1°$, will they still win the jersey? (If it helps, a hockey puck is 3 inches in diameter.)

Inverse Sine, Inverse Cosine, Inverse Tangent

Okay, we now know how to find the length of a side if we know an angle and another side. But what if we know the lengths of the sides and want to determine a missing angle? Rather than carefully making a scale drawing and using a protractor, we can use a calculator.

Your calculator's SIN, COS, and TAN keys take an angle measure for the input and give a ratio (in decimal form) as the output. Your calculator's SIN^{-1}, COS^{-1}, and TAN^{-1} keys allow you to work backwards; they take a ratio for the input and give an angle measure for the output.

Exercises

Use your calculator to determine each of the following angle measures (rounded to the nearest hundredth of a degree, if necessary).

24. $\sin^{-1}\left(\frac{3}{5}\right)$

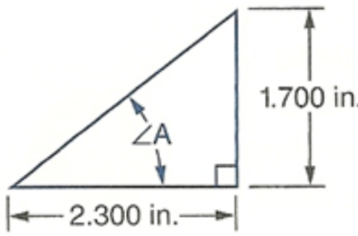
25. $\cos^{-1}\left(\frac{1}{2}\right)$

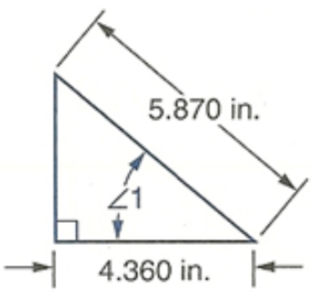
26. $\tan^{-1}\left(\frac{4}{3}\right)$

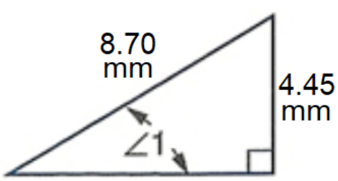
Inverse Trigonometry Applications

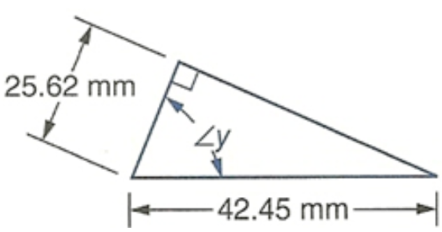
Exercises

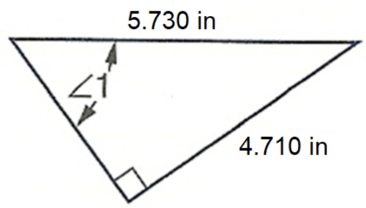
Determine the measure of the marked angle. Round each answer to the nearest hundredth of a degree.

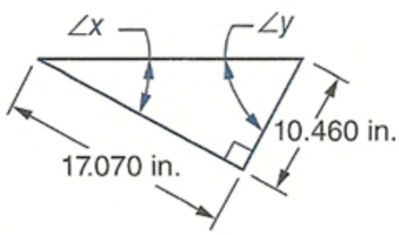
27. 

28. 

29. 

30. 

31. 

32. 

33. A rule of thumb for ladder safety is that the vertical height where the ladder touches the wall should be four times the horizontal distance from the base of the ladder to the wall.[1] To the nearest degree, what is the angle between the ladder and the floor in this scenario?

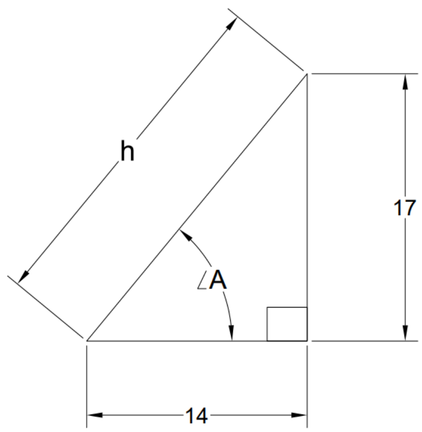
34. The entrance to a public building is 2 feet higher than the courtyard in front of the entrance. A ramp is being constructed, continuous with no switchbacks or level platforms, with a surface length of 25 feet. ADA regulations[2] limit the angle of elevation to a maximum of approximately $4.75°$. Does the ramp's design meet ADA standards?

Photo by UNDP in Europe and Central Asia on flickr.

Using Multiple Methods

In Module 18, we used the Pythagorean Theorem, $a^2 + b^2 = c^2$, to determine the length of an unknown side in a right triangle. We can use the Pythagorean Theorem in conjunction with trigonometry to solve a problem in more than one way or to double-check our results.

Exercises



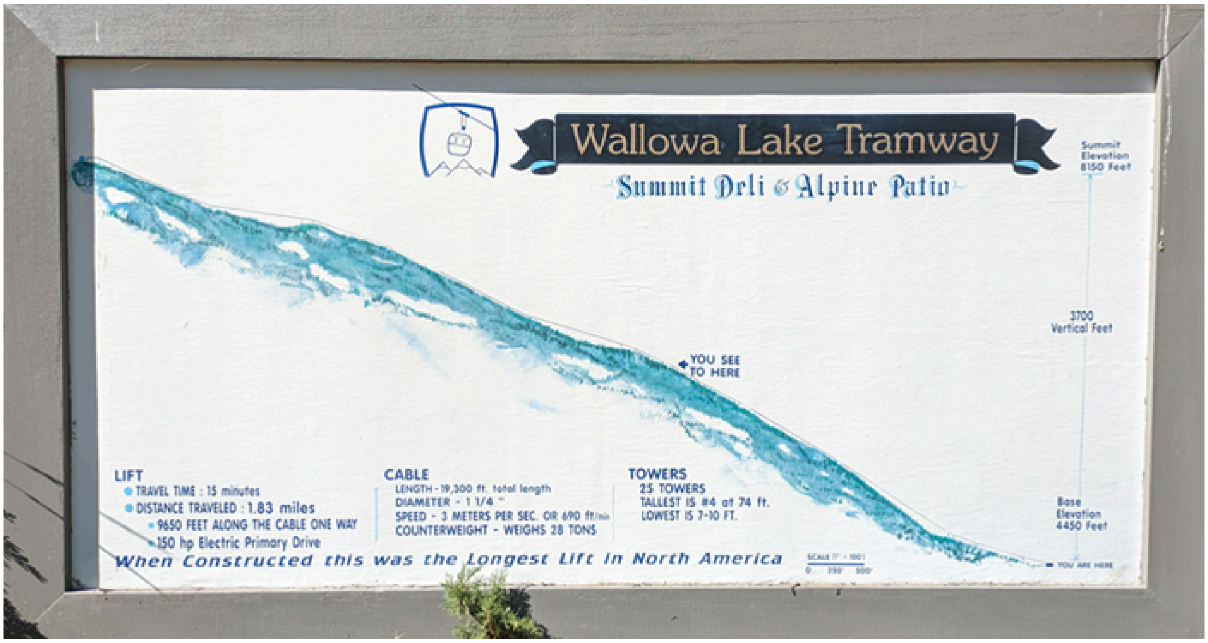
35. First, use inverse tangent to determine the value of A. Round to the nearest tenth of a degree.

36. Next, use sine with $\angle A$ to determine the value of h. Round to the nearest tenth.

37. Next, use cosine with $\angle A$ to determine the value of h. Round to the nearest tenth.

38. Finally, use the Pythagorean Theorem to determine the value of h. Round to the nearest tenth.

39. Compare your answers for Exercises 36-38. Are they the same or are they different?



40. The Wallowa Lake Tramway in Eastern Oregon carries passengers along a $9,650$ ft cable, with a vertical rise of $3,700$ ft. Although the cable is not straight along its entire length, let's assume that it is. To the nearest degree, what is the angle of elevation of the cable?

41. What is the approximate horizontal run covered by the tramway cable?

Exercise Answers

Notes

1. Source: https://www.americanladderinstitute.org/page/Ladders101
2. This is implied at https://www.ada-compliance.com/ada-compliance/ada-ramp but not directly stated. We'll verify this angle measure in the next module.

[32]

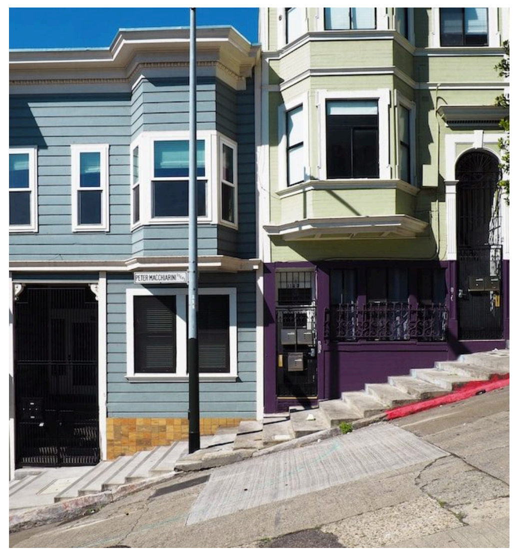
Slope

Bob Brown and Morgan Chase

You may use a calculator throughout this module.

The ***slope*** of a surface is a measure of its steepness. In some cases, such as a walkway or ramp or street, a shallow slope is safer than a steep slope. In other cases, such as a roof, a steep slope may be preferred because it allows rainwater or accumulated snow to move off the roof surface more easily than a shallow slope.[1]

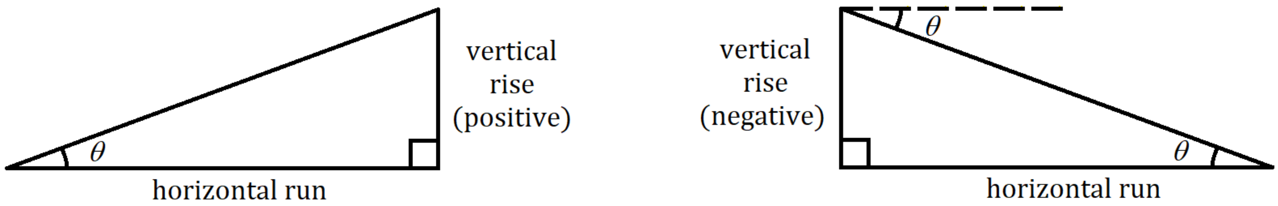
We will first look at slope in terms of vertical and horizontal distances, and we will then look at slope in terms of angles. Come on, let's hit the slopes!



This section of Kearney Street in San Francisco is too steep for a sidewalk and has stairs instead. Photo by Marcus Lenk on Unsplash.

Slope as a Ratio

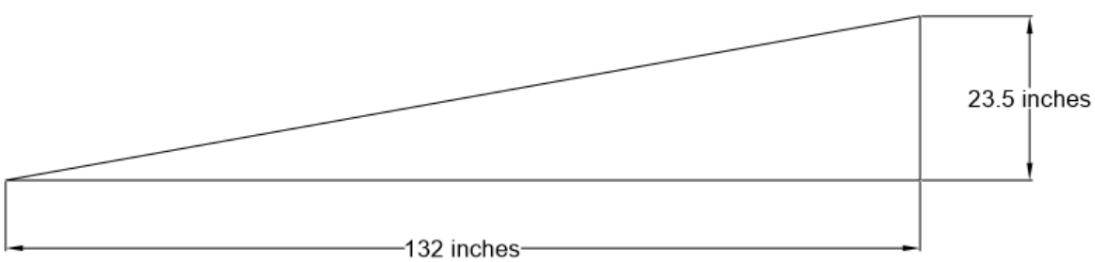
Slope is defined as the ratio of the vertical rise to the horizontal run.



$$\text{slope} = \frac{\text{vertical rise}}{\text{horizontal run}}$$

- A line that is increasing in height has a positive slope.
- A line that is decreasing in height has a negative slope.[2]
- The slope of a horizontal line is 0; it is not increasing or decreasing.

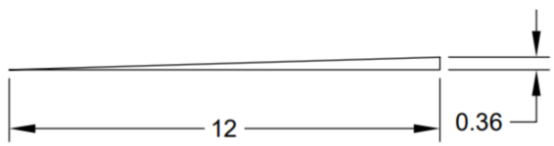
A slope may be expressed as a ratio, a decimal, or a percent grade. For example, consider this loading ramp with a vertical rise of 23.5 inches and a horizontal run of 132 inches.



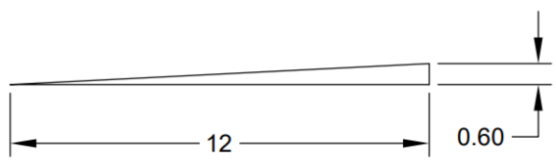
As a ratio, the slope is 23.5 : 132. Dividing the rise by the run gives a decimal value of approximately 0.18. Moving the decimal point two places to the right gives a grade of approximately 18%.

Exercises

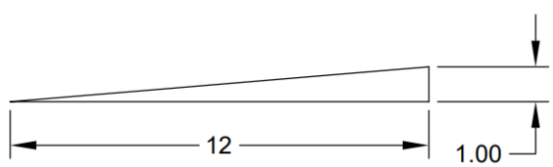
Express each slope in three ways: as a ratio, a decimal, and a percent grade.

1. 

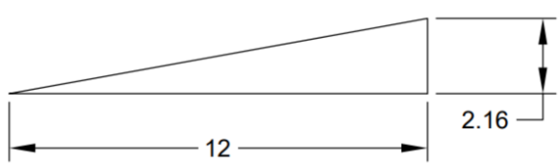
(This is the preferred maximum slope for sidewalks.)

2. 

(This is the maximum allowed slope for sidewalks.)[3]

3. 

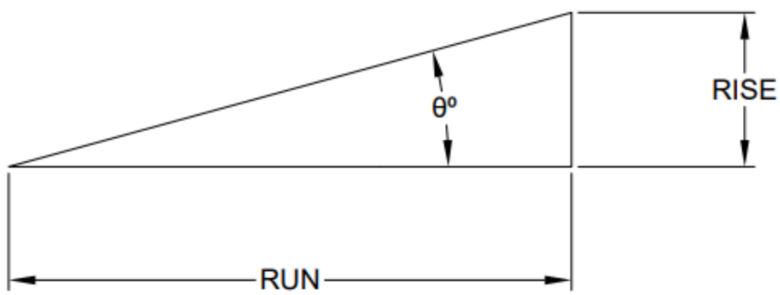
(This is the maximum allowed slope for ADA-accessible ramps.)[4]

4. 

(This is the preferred maximum slope for open pit mines.)

Slope as an Angle

The steepness of a line may also be described by its angle of elevation above the horizontal (or its angle of depression below the horizontal).



Notice that the vertical rise is the side ***opposite*** the angle, and the horizontal run is the side ***adjacent*** to the angle. Therefore, trigonometry tells us that $\text{tangent} = \frac{\text{opposite}}{\text{adjacent}} = \frac{\text{rise}}{\text{run}}$. The tangent of the angle is equal to the slope.

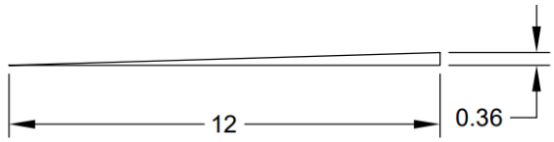
$$\tan\,\theta = \frac{\text{rise}}{\text{run}}$$

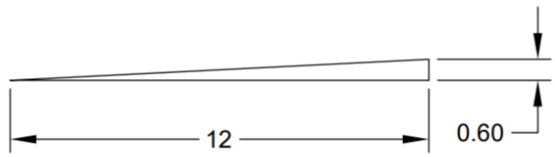
Or, thinking about it in reverse, the inverse tangent of the slope is the angle.

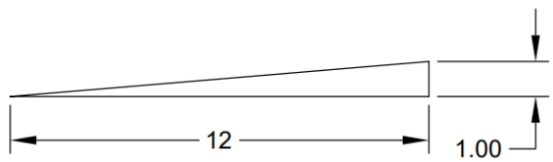
$$\tan^{-1}\left(\frac{\text{rise}}{\text{run}}\right) = \theta$$

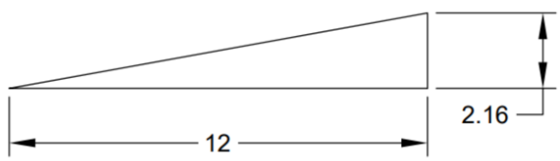
Exercises

Determine the angle of elevation for each slope. Round to the nearest hundredth of a degree.

5. 

6. 

7. 

8. 

For proper drainage, the ground around a building should slope downwards, away from the building.

9. The minimum downward grade of the ground is 1%. Assuming this grade, if a point on the ground is 50 feet horizontally from the base of the house, how much lower is the ground at that point?

10. The preferred minimum downward grade of the ground is 2%. Assuming this grade, if a point on the ground is 50 feet horizontally from the base of the house, how much lower is the ground at that point?

11. The maximum acceptable downward grade of the ground is 10%. Assuming this grade, if a point on the ground is 50 feet horizontally from the base of the house, how much lower is the ground at that point?

---

12. A motion sensor needs to be installed on the outside of a warehouse door 12 feet above the ground. The sensor should go off if anyone approaching the warehouse gets within 20 feet of the door. What angle from vertical should

the sensor point away from the building to detect someone at the appropriate distance from the warehouse? Round your answer to the nearest degree.

13. A ramp is being constructed to the entrance to a public building that is 2.5 feet higher than the level courtyard in front of the entrance. Assuming that the ramp will be continuous with no switchbacks or level platforms,[5] what is the minimum horizontal distance required for the ramp so that it will comply with ADA regulations?

Photo by Forest Service Pacific Northwest Region on flickr.

Exercise Answers

Notes

1. Source: https://www.nachi.org/roof-slope-pitch.htm
2. We won't worry about negative slopes in this textbook, because we can always express a negative slope using a word like "decrease", "decline", or "depression" in combination with a positive number.
3. Source: https://www.ada-compliance.com/ada-compliance/403-walking-surfaces
4. Source: https://www.ada-compliance.com/ada-compliance/405-ramps
5. If you were curious, this is the maximum rise allowed for this situation; a rise of more than 2.5 feet would require a switchback or a level platform.

[33]

Non-Right Triangle Trigonometry

You may use a calculator throughout this module.

Basic trigonometry applies to right triangles, but there are two useful formulas that can be used with non-right triangles; these are known as the Law of Sines and the Law of Cosines.

The Law of Sines allows you to use a proportion to solve for a missing value in a triangle.

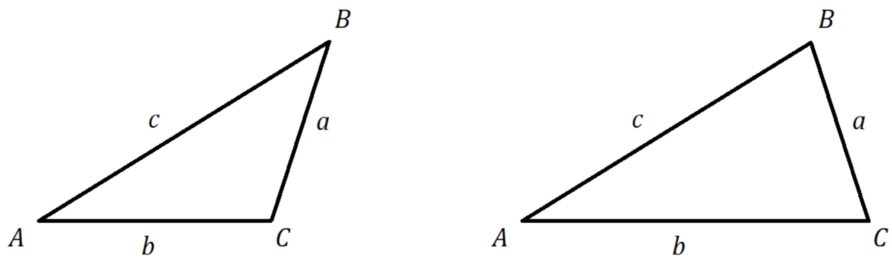
The Law of Cosines is essentially the Pythagorean Theorem with an extra twist that makes it work with any kind of triangle.

These formulas have applications in many fields, most obviously in surveying and forestry where distances cannot be measured directly.

Your author among the California Redwoods.

Law of Sines

Suppose we have a triangle with its vertices labeled A, B, C, and the sides opposite each vertex labeled a, b, and c.

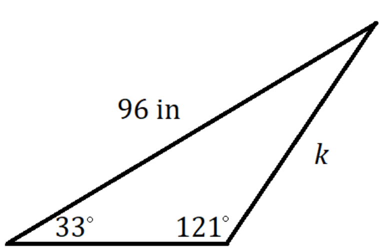


The Law of Sines is useful when you are dealing with two sides and two angles; if you know three of those values, you can use the formula to figure out the fourth value. Depending on which sides and angles you know, you'll use one of these three versions:

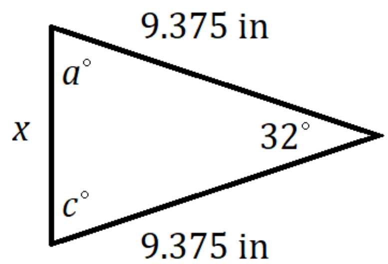
$$\frac{\sin\ A}{a} = \frac{\sin\ B}{b} \qquad \frac{\sin\ B}{b} = \frac{\sin\ C}{c} \qquad \frac{\sin\ C}{c} = \frac{\sin\ A}{a}$$

Exercises

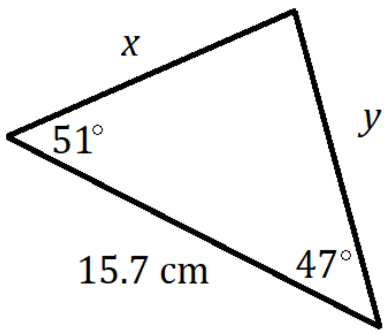
Determine the unknown value(s) in each triangle.



1.

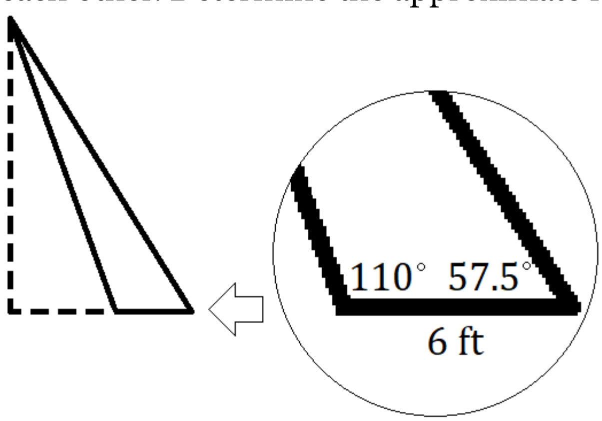


2.



3.

4. A utility pole is supported by two guy wires as shown in the figure below. The shorter wire makes a $110°$ angle with the sidewalk, the longer wire makes a $57.5°$ angle with the sidewalk, and the wires meet the sidewalk a distance of 6 feet from each other. Determine the approximate length of each wire.

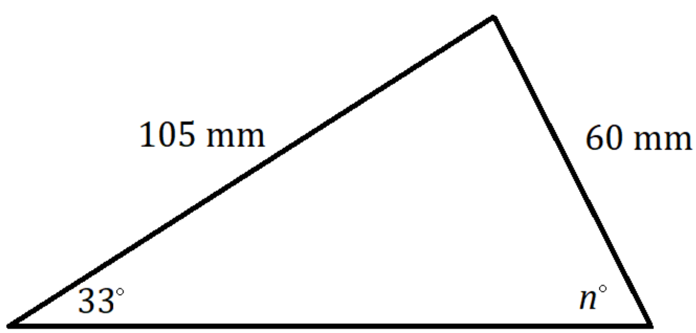


When we need to determine an angle measure, we sometimes have a situation with two possible results: the angle could be acute or obtuse. (For this ambiguous situation to arise, we must know the lengths of two sides and the measure of an acute angle that is *not* between those two sides.) The inverse sine function on a calculator will always be programmed to give an acute angle measure for the result. If it is clear that the angle should be obtuse, simply subtract the calculator's result from 180 degrees.

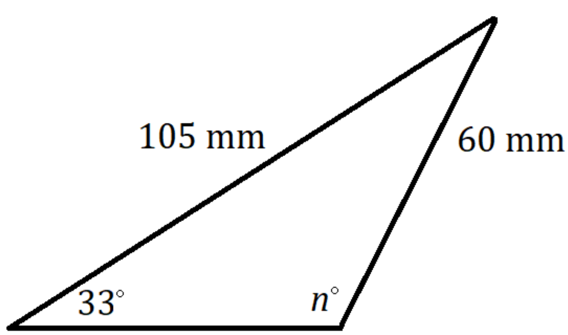
Exercises

Determine the unknown angle measure in each triangle.

5. Assume that n represents an acute angle.

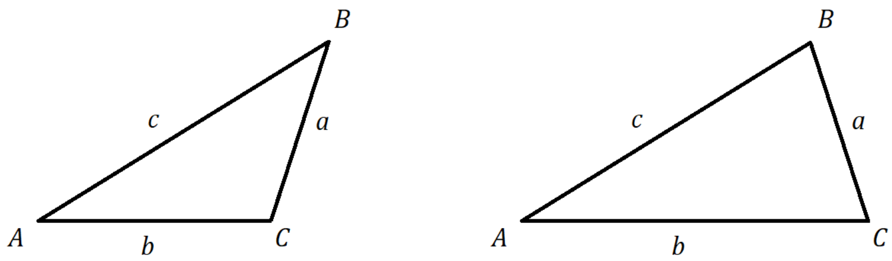


6. Assume that n represents an obtuse angle.



Law of Cosines

Again, suppose we have a triangle with its vertices labeled A, B, C, and the sides opposite each vertex labeled a, b, and c.



The Law of Cosines is useful when you are dealing with three sides and one angle; if you know three of those values, you can use the formula to figure out the fourth value.

If you're trying to determine one of the side lengths, use this version.

$$c = \sqrt{a^2 + b^2 - 2ab \cos C}$$

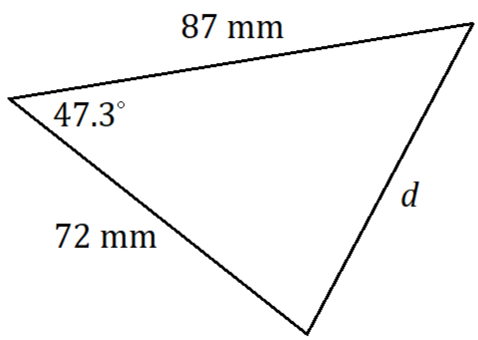
If you're trying to determine an angle measure, use this version.

$$C = \cos^{-1}\left(\frac{a^2+b^2-c^2}{2ab}\right)$$

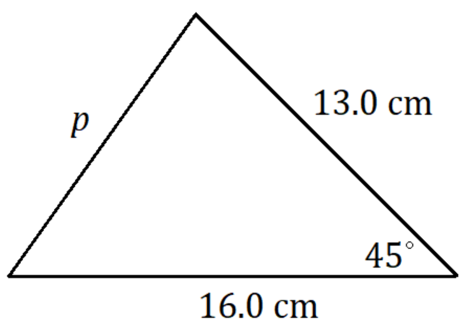
In either case, the side you name c must be opposite the angle you name C.

Exercises

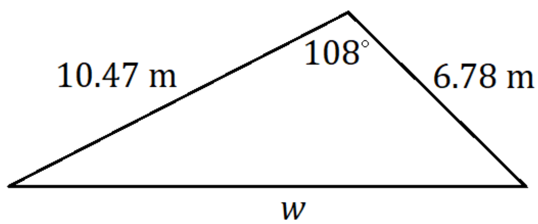
Determine the unknown value in each triangle.



7.

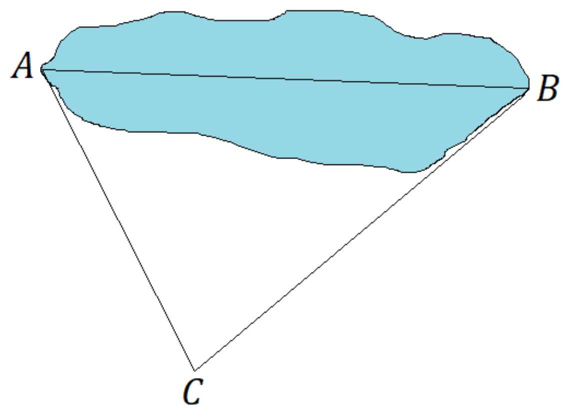


8.

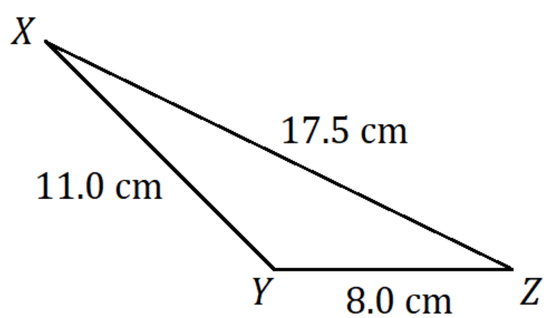


9.

10. Amateur surveyors have determined that the distance from C to A is 53 meters, the distance from C to B is 75 meters, and the measure of $\angle C$ is $77°$, as shown in the figure below. What is the length of the pond?



11. Determine the measure of each angle.



Exercise Answers

[34]

Radian Measure

You may use a calculator as needed in this module.

In everyday life, we measure angles in degrees. However, degrees are an arbitrary unit of measure; why is a right angle 90 degrees and a full circle 360 degrees?[1]

Another unit that is used in surveying is the *gradian*; a right angle is equal to 100 gradians. Using gradians makes some calculations easier, but this is still an arbitrary unit of measure.

In this module, we will focus on a third unit of angle measure known as ***radians***.

Radian Measure

To explain radian measure, we can visualize a circle. Next, let's measure a length along the circumference of a circle that is equal to the radius. Finally, let's draw an angle with its vertex at the center of the circle, intersecting the circle at the endpoints of this arc. The measure of this angle is defined to be 1 radian.

In fewer words, a radian is the measure of the central angle that forms an arc equal to the circle's radius.

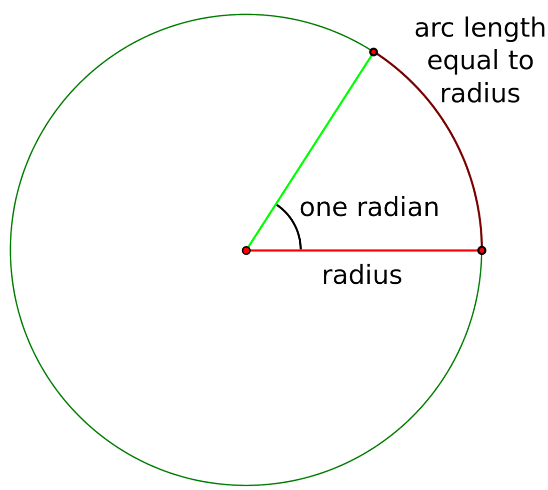
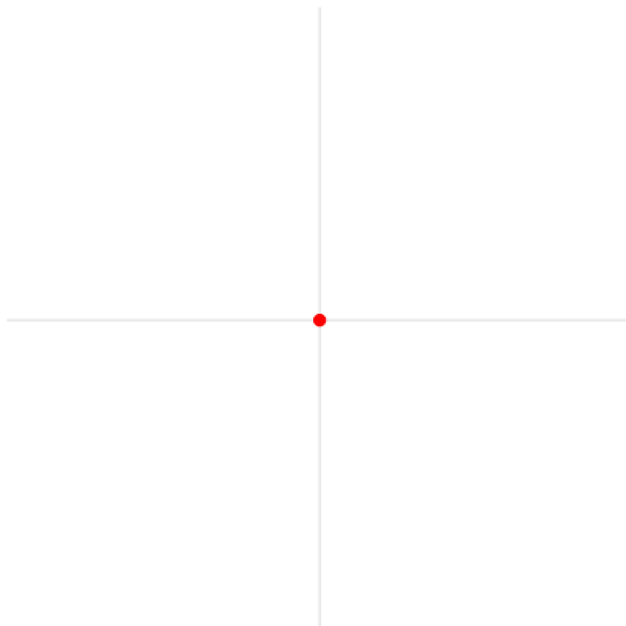


Image by Adrignola on Wikimedia Commons.

Because the circumference of a circle is 2π times the radius, there are 2π radians in one full circle, and therefore 2π radians are equivalent to $360°$.

$$2\pi \text{ rad} = 360°$$

$$\pi \text{ rad} = 180°$$



Animation by Lucas Vieira on Wikimedia Commons.

We can convert between degrees and radians using the conversion factor $\pi \text{ rad} = 180°$.

Exercises

Convert each degree measure into radian measure. Leave your answers in terms of π.

1. $30°$
2. $45°$
3. $60°$

4. $90°$
5. $120°$
6. $225°$
7. $240°$
8. $270°$

Convert each radian measure into degree measure.

9. $\frac{\pi}{12}$
10. $\frac{3\pi}{4}$
11. $\frac{5\pi}{6}$
12. $\frac{7\pi}{4}$
13. 1

Exercise Answers

Notes

1. See https://en.wikipedia.org/wiki/Degree_(angle)#History for some theories.

Exercise Answers

Module 1: Order of Operations

1. 7
2. 13
3. 7
4. 13
5. 2
6. 8
7. 18
8. 6
9. 25
10. 49
11. 80
12. 31
13. 28
14. 67
15. 22
16. 4

17. 160

18. 19

19. 2

20. 12

21. 40

22. 200

23. 2

24. 14

25. $9 \cdot 2 + 30 = 48^{\circ}\mathrm{F}$

26. $(72 - 30) \div 2 = 21^{\circ}\mathrm{C}$

Module 2: Negative Numbers

1. 5

2. 5

3. -15

4. -22

5. 4

6. -4

7. -9

8. 9

9. $18^{\circ}\mathrm{F}$

10. 3

11. -200

12. 3

13. -3

14. -7

15. -7

16. 7

17. 7

18. 3

19. -3

20. 55

21. 55

22. $11,123$ ft

23. 543 ft

24. -12

25. -40

26. 18

27. 21

28. 4

29. -8

30. 16

31. -32

32. -7

33. -4

34. 9

35. 0

36. 0

37. undefined

38. 19

39. -73

40. 1

41. -6

42. -8

43. 40

Module 3: Decimals

If you've seen Modules 5 & 6, don't worry about accuracy or precision on these exercises.

1. 90.23
2. 7.056
3. 16.55
4. 184.015
5. 8.28
6. 15.756
7. $4,147$
8. 414.7
9. 41.47
10. 4.147

11. $65,625$

12. 65.625

13. 6.5625

14. $\$656.25$

15. 243.5

16. $2,435$

17. $243,500$

18. 24.35

19. $6,000$

20. $6,380$

21. 0.71

22. 0.715

23. $\$3.67$ per month

24. 7.5 miles per hour

Module 4: Fractions

1. $\frac{11}{30}$

2. $\frac{19}{30}$

3. $\frac{12}{15}$

4. $\frac{8}{12}$

5. 7

6. 1

7. 0

8. undefined
9. $\frac{3}{4}$
10. $\frac{5}{3}$
11. 2
12. $\frac{1}{2}$
13. $\frac{5}{12}$
14. 1
15. at least 45 questions
16. 16
17. $\frac{3}{5}$
18. 6 scoops
19. A requires $\frac{1}{12}$ cup more than B
20. $\frac{1}{2}$ of the pizza
21. $\frac{2}{3}$ more
22. $\frac{5}{8}$ inches combined
23. $\frac{1}{8}$ inches difference
24. $\frac{7}{12}$ combined
25. $\frac{1}{12}$ more
26. 2.75
27. 0.35
28. $0.\overline{5}$ or $0.555...$
29. $1.\overline{63}$ or $1.636363...$

30. $11\frac{1}{2}$
31. $4\frac{2}{3}$
32. $\frac{11}{5}$
33. $\frac{20}{3}$
34. $10\frac{3}{8}$
35. $4\frac{7}{8}$
36. $8\frac{1}{6}$
37. $1\frac{5}{6}$ cup

Module 5: Accuracy and Significant Figures

1. exact value
2. approximation
3. exact value
4. approximation
5. exact value
6. approximation
7. three significant figures
8. four significant figures
9. five significant figures
10. two significant figures
11. three significant figures
12. four significant figures
13. two significant figures; the actual value could be anywhere between $28,500$ and

$29,500$

14. three significant figures; the actual value could be anywhere between $28,950$ and $29,050$

15. four significant figures; the actual value could be anywhere between $28,995$ and $29,005$

16. five significant figures; the actual value could be anywhere between $28,999.5$ and $29,000.5$

17. $51,800$

18. $51,840$

19. 4.3

20. 4.28

21. $14,000$

22. $14,\overline{0}00$

23. 2.6

24. 2.60

25. $29,000 \text{ ft}$

26. $29,\overline{0}00 \text{ ft}$

27. $29,030 \text{ ft}$

28. $29,032 \text{ ft}$

29. $29,031.7 \text{ ft}$

30. 107

31. 640

32. 14.4

33. 12

34. $\$23$

Module 6: Precision and GPE

1. thousands
2. hundreds
3. tens
4. thousandths
5. ten thousandths
6. hundred thousandths
7. $82,000$
8. $82,\bar{0}00$
9. $82,0\bar{0}0$
10. 0.6
11. 0.60
12. 0.600
13. 39.3 lb
14. 39 lb
15. $\$9,800$
16. $\$8\bar{0}0$
17. thousands place; the nearest $1,000$ people
18. ± 500 people
19. hundred thousandths place; the nearest 0.00001 in
20. ± 0.000005 in

21. hundredths place; the nearest 0.01 mil
22. ± 0.005 mil
23. $30\overline{0}$ miles
24. ones place; the nearest 1 mile
25. ± 0.5 mi
26. ones place; the nearest 1 minute
27. ± 0.5 min
28. two sig figs
29. three sig figs
30. 55 mi/hr
31. When dividing, we must round the result based on the accuracy; i.e., the number of significant figures.

Module 7: Formulas

1. $\$2.46$
2. $\$4.14$
3. $\$17.80$
4. $\$24.80$
5. $\$3.80$
6. 6 representatives
7. 10 representatives
8. 52 representatives
9. 8 electoral votes

10. 12 electoral votes
11. 54 electoral votes
12. $\approx 115°$F; the official record high in the city was 116°F.
13. $37°$C
14. $-0.4°$F
15. $200°$C
16. 90 mm Hg
17. around 107 mm Hg
18. 70 in
19. 74 in
20. yes
21. no; too large
22. no; too small
23. yes

Module 8: Perimeter and Circumference

1. 70 ft
2. 58 cm
3. 28 cm
4. 104 ft
5. 130 ft
6. 56 ft
7. 24 in

8. 20 cm

9. 28.3 in

10. 18.8 cm

11. 44.0 ft

12. 53 m

Module 9: Percents Part 1

1. 47%

2. 53%

3. $\frac{71}{100}$

4. $\frac{1.3}{100} = \frac{13}{1000}$

5. $\frac{0.04}{100} = \frac{1}{2500}$

6. $\frac{106}{100} = \frac{53}{50}$

7. 0.71

8. 0.013

9. 0.0004

10. 1.06

11. 23%

12. 7%

13. 8.5%

14. 250%

15. 28%

16. 12.5%

17. 31.5

18. 22.5

19. 67.5

20. 100

21. 38.6

22. 2.25

23. $\$9.35$

24. $\$119.32$

Module 10: Ratios, Rates, Proportions

1. $\frac{3}{8}$

2. $\frac{105 \text{ mi}}{2 \text{ hr}}$

3. $\frac{52.5 \text{ mi}}{1 \text{ hr}}$ or 52.5 miles per hour

4. $\approx \$0.335/\text{oz}$, or $33.5¢/\text{oz}$

Carmella Creeper parties like it's 1995.

5. $\approx \$0.281/\text{oz}$, or $28.1¢/\text{oz}$

6. $\approx \$0.281/\text{oz}$, or $28.1¢/\text{oz}$

7. $\frac{3}{4} = \frac{3}{4}$; true

8. $\frac{2}{3} \neq \frac{4}{5}$; false

9. $168 = 168$; true

10. $200 \neq 240$; false

11. $70 \neq 60$; false

12. $20 = 20$; true

13. $x = 12$

14. $n = 5$

15. $k = 4$

16. $w = 10$

17. $x = 10.4$

18. $m = 2.0$

19. 256 miles

20. 20 hours

21. ≈ 50 miles (rounding to one sig fig seems like a good idea here)

22. 190 pixels wide

Module 11: Scientific Notation

1. Earth's mass is larger because it's a 25-digit number and Mars' mass is a 24 -digit number, but it might take a lot of work counting the zeros to be sure.

2. Earth's mass is about ten times larger, because the power of 10 is 1 higher than that of Mars.

3. A chlorine atom's radius is larger because it has 9 zeros before the significant digits begin, but a hydrogen atom's radius has 10 zeros before the significant digits begin. As above, counting the zeros is a pain in the neck.

4. The chlorine atom has a larger radius because its power of 10 is 1 higher than that of the hydrogen atom. (Remember that -10 is larger than -11 because -10 is farther to the right on a number line.)

5. $\$100,000$; 1.00000×10^5

6. $\$999,999$; 9.99999×10^5

7. 1.234×10^3

8. 1.02×10^7

9. 8.70×10^{-4}

10. 7.32×10^{-2}

11. $35,000$

12. $90,120,000$

13. 0.00825

14. 0.000014

15. 8×10^7

16. 3.5×10^{13}

17. 6×10^{-5}

18. 4.8×10^5

19. $1,260$ people per square mile

20. 250 people per square mile

21. the proton's mass is roughly $1,830$ or 1.83×10^3 times larger

22. 335.9×10^6; 3.359×10^8

23. 8.020×10^9; 8.020×10^9

24. 33.9×10^{12}; 3.39×10^{13}

25. $\approx \$101,000$ per person

Module 12: Percents Part 2 and Error Analysis

1. 93% or 93.3%
2. 37.5%
3. $\$2,500$
4. 720
5. 93% or 93.3%
6. 37.5%
7. $\$2,500$
8. 720
9. 44.8% or 45%
10. 50 grams of added sugars is the recommended daily intake for a $2,000$ calorie diet.
11. 56% increase
12. 10.1% sales tax
13. 36% decrease
14. 2.7% decrease
15. $0.1875 \div 25 \approx 0.75\%$
16. $0.13 \div 10.8 \approx 1.2\%$
17. 4.806 g; 5.194 g
18. 3.88%
19. 5.443 g; 5.897 g

20. 4.00%

Module 13: The US Measurement System

We generally won't worry about significant figures in these answers; we'll probably say "2 miles" even if "2.000 miles" is technically correct.

1. 54 in

2. 54 ft

3. 36 in

4. $1,760$ yd

5. $14\frac{2}{3}$ ft or 14 ft 8 in

6. 15 yd

7. 2 mi

8. 30 yd

9. 40 oz

10. $2,400$ lb

11. 18.75 lb

12. $32,000$ oz

13. 48 fl oz

14. 7 pt

15. 8 pt

16. 5 c

17. 1.25 gal

18. 64 fl oz

19. 3 lb 7 oz
20. 7 c 3 fl oz
21. 15 ft 2 in
22. 4 t 500 lb
23. 20 lb or 20 lb 0 oz combined
24. 3 lb 2 oz heavier
25. 9 ft 1 in combined
26. 1 ft 5 in longer

Module 14: The Metric System

1. 5 m
2. 28 cm
3. 3.8 km
4. 1.6 m
5. 160 cm
6. 3 mm
7. 536 cm
8. 5,360 mm
9. 1,609 m
10. 160,900 cm
11. 0.297 m
12. 29.7 cm
13. 0.828 km

14. 82.8 dam
15. 100 g
16. 80 kg
17. 500 mg
18. $2,000$ kg
19. $2,270$ g
20. $2,270,000$ mg
21. $6,500$ cg
22. $65,000$ mg
23. 0.065 kg
24. 9.5 cg
25. 0.095 g
26. 50 L
27. 30 mL
28. 0.5 L
29. 10.5 dL
30. $1,750$ mL
31. 0.25 L
32. they are equal in size
33. about 11 to 1
34. 4 bottles; this is easier if you know that a 500-milliliter bottle of Mexican Coke is called a *medio litro*.

Doce litros de Coca-Cola

Module 15: Converting Between Systems

We may round some of these answers to three significant figures even if the given number has fewer than three sig figs. On the other hand, you'll see that some of these answers include a critique of a manufacturer's decision to round numbers a certain way.

1. 31 mi
2. 183 cm
3. 164.0 ft
4. yes, $4 \text{ in} = 101.6 \text{ mm}$ according to the conversion, but is it really 4.000 in to begin with? Rounding the result to 100 mm or 102 mm seems reasonable.
5. 20 in converts to around 50.8 cm, and 50.0 cm converts to around 19.7 in. It looks like somebody used the conversion $1 \text{ in} = 2.5 \text{ cm}$, which is fine if you're estimating but not if you're going to report a number to three sig figs.
6. not exactly but they're pretty close; the error is around 0.3%.
7. yes; using either conversion gives a result of 294.8 kg, rounded to the nearest tenth. However, it doesn't make sense for this to be accurate to four sig figs. It would be best to round to 295 or perhaps even $3\bar{0}0$ kg.
8. not exactly; using 1 kg $\approx$ 2.205 lb gives a result of $1,473.9$ kg, and using 1 lb

≈ 0.4536 kg gives a result of $1,474.2$ kg. Again, the accuracy here doesn't make sense, especially when you consider that the numbers on the box don't agree with each other: $294.8 \cdot 5 \neq 1,474.1$.

9. 11 lb
10. $\approx 230 \text{ g}$
11. $\approx 110 \text{ g}$
12. $\approx 1.8 \text{ oz}$
13. about $\$1.09$
14. about $\$0.99$
15. $\approx 1.3 \text{ gal}$
16. 355 mL
17. 3.4 fl oz
18. no matter which conversion you use, the result should round to 22.7 liters.
19. a bit less than 600 km
20. 11 km/L

Module 16: Other Conversions

We will generally round these answers to three significant figures; your answer may be slightly different depending on which conversion ratio you used.

1. $525,600$ min; if you're familiar with the musical *Rent*, then you already knew the answer.
2. this is roughly 31.7 years, which is indeed possible
3. 37.6 km/hr
4. 23.3 mi/hr

5. $1,770$ mi/hr
6. 29.5 mi in 1 min
7. 20.3 min
8. 0.17 mi/gal
9. 5.8 gal/mi
10. 171 gal in 1 min
11. the capacity increased by a factor of 14.4
12. 4 times greater
13. $1,200$ megawatts per home
14. 1 watt per gallon
15. $2,500$ times more powerful
16. 0.4 ms, 0.04 ms, 0.004 ms; 400 µs, 40 µs, 4 µs
17. the ratio of the wavelengths of red and infrared is 7 to 100;
    the ratio of the wavelengths of infrared and red is around 14 to 1
18. this is equivalent to $2,500$ chest x-rays

Module 17: Angles

1. right angle
2. obtuse angle
3. reflexive angle
4. straight angle
5. acute angle
6. $a = 127°$; $b = 53°$; $c = 127°$

7. $27°$
8. $97°$
9. $23°$ each
10. $45°$ each
11. $60°$ each
12. $A = 61°$; $B = 80°$; $C = 39°$
13. $18.9111°$
14. $155.6808°$
15. $34.1924°$
16. $29°58'30''$
17. $31°8'15''$
18. $76°20'48.1''$

Module 18: Triangles

1. right isosceles triangle
2. obtuse scalene triangle
3. acute equilateral triangle (yes, an equilateral triangle will always be acute)
4. $w = 35$ ft
5. $x = 8$ cm; $y = 10.5$ cm
6. $d = 268$ ft
7. $n = 55$ cm
8. this is a right triangle, because $5^2 + 12^2 = 13^2$.
9. this is not a right triangle, because $8^2 + 17^2 \neq 19^2$.

10. 7.07

11. 17.20

12. 30.71

13. 10 ft

14. 15 ft

15. 12.3 cm

16. 1.8 cm

Module 19: Area of Polygons and Circles

We may occasionally include extra sig figs in these answers so you can be sure that your answer matches ours.

1. $20\ \text{cm}^2$

2. $16\ \text{cm}^2$

3. $4.86\ \text{m}^2$

4. $12.25\ \text{ft}^2$

5. $120\ \text{in}^2$

6. $360\ \text{m}^2$

7. $210\ \text{ft}^2$

8. $126\ \text{cm}^2$

9. $38.5\ \text{cm}^2$

10. $204\ \text{ft}^2$

11. $2,160\ \text{in}^2$

12. $36\ \text{m}^2$

13. 124 cm^2

14. 192 cm^2

15. 28.3 cm^2

16. 220 m^2

17. 154 ft^2

18. 63.6 in^2

Module 20: Composite Figures

1. 64 ft

2. 189 ft^2

3. 590 cm^2; the area of the rectangle is 800 cm^2 and the areas of the triangles are 70 cm^2 and 140 cm^2.

4. 590 cm^2; hey, that's what we got for #3!

5. 148 m

6. 940 m^2

7. Based on the stated measurements, the distance around the track will be 401 meters, which appears to be 1 meter too long. In real life, precision would be very important here, and you might ask for the measurements to be given to the nearest tenth of a meter.

8. around $9,620 \text{ m}^2$

9. $1,960 \text{ cm}^2$

10. 178.5 cm

11. 29 ft^2

12. 47 ft

13. around 21.5% (Hint: Make up an easy number for the side of the square, like 2 or 10.)

14. around 36.3% (Hint: The diagonals of the square are equal to the circle's diameter.)

Module 21: Converting Units of Area

We may occasionally include extra sig figs in these answers so that you can be sure that your answer matches ours.

1. 162 ft^2
2. 162 ft^2
3. 7 ft^2
4. 7 ft^2
5. $7,776 \text{ in}^2$
6. 8.3 ac
7. $180,000 \text{ cm}^2$
8. $180,000 \text{ cm}^2$
9. 623.7 cm^2
10. 623.7 cm^2
11. 6 m^2
12. 4 ha
13. 376 km^2
14. $2,\overline{0}00 \text{ ha}$, rounded to two sig figs
15. 603 cm^2
16. $75,300 \text{ ft}^2$, rounded to three sig figs

17. 154 in^2

18. 38.5 in^2

19. 4 to 1; if we double the linear measurement, we get four times the area.

20. 54 cm^2

21. 6 cm^2

22. 9 to 1; if we triple the linear measurement, we get nine times the area.

Module 22: Surface Area of Common Solids

1. 36 cm^2
2. 76 cm^2
3. 471 cm^2
4. 628 cm^2
5. 616 cm^2
6. 380 in^2

Module 23: Area of Regular Polygons

All answers have been given to two or three significant figures.

1. $6,900 \text{ in}^2$
2. 94 cm^2
3. 750 in^2
4. 751 mm^2
5. 480 cm^2
6. 110 m^2

7. $280\ \mathrm{mm}^2$ (the area of the circle $\approx 1,660\ \mathrm{mm}^2$ and the area of the hexagon is $1,380\ \mathrm{mm}^2$)

Module 24: Volume of Common Solids

1. $40\ \mathrm{cm}^3$
2. $531\ \mathrm{cm}^3$
3. $45\ \mathrm{cm}^3$
4. $350\ \mathrm{cm}^3$
5. $520\ \mathrm{cm}^3$
6. $22,600\ \mathrm{ft}^3$
7. $3.7\ \mathrm{mm}^3$
8. $1,440\ \mathrm{cm}^3$
9. $697\ \mathrm{in}^3$ or $7\overline{0}0\ \mathrm{in}^3$
10. $22.7\ \mathrm{cm}^3$ (the cylinder's volume $\approx 14.1\ \mathrm{cm}^3$ and the hemisphere's volume $\approx 8.6\ \mathrm{cm}^3$.)
11. $37.6\ \mathrm{ft}^3$ (the cylinder's volume $\approx 29.45\ \mathrm{ft}^3$ and the two hemispheres' combined volume $\approx 8.18\ \mathrm{ft}^3$)
12. $37.6\ \mathrm{ft}^3 \approx 282\ \mathrm{gal}$, which is more than $250\ \mathrm{gal}$.

Module 25: Converting Units of Volume

1. the result is very close to 1 cubic yard:
   $(112\ \mathrm{in} \cdot 14\ \mathrm{in} \cdot 10\ \mathrm{in}) \cdot 3\ \mathrm{crates} = 47,040\ \mathrm{in}^3 \approx 1.01\ \mathrm{yd}^3$
2. this estimate is also 1 cubic yard: $(9\ \mathrm{ft} \cdot 1\ \mathrm{ft} \cdot 1\ \mathrm{ft}) \cdot 3\ \mathrm{crates} = 27\ \mathrm{ft}^3 = 1\ \mathrm{yd}^3$
3. around 60 gallons

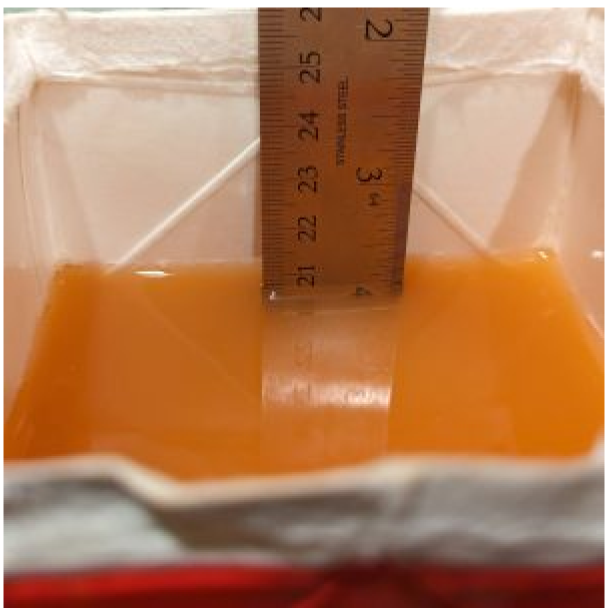
4. yes, the can is able to hold 12 fluid ounces; the can's volume is roughly $23.3 \text{ in}^3 \approx 12.9 \text{ fl oz}$.

5. 5 gallons

6. around $2,300$ to $2,400$ liters; a calculator says $2,356$ liters which should technically be rounded up to $2,400$ liters, but it would be reasonable to round down to $2,300$ liters instead if you considered the volume of the benches and the fact that the sides might slope inwards near to bottom of the tub.

7. the rectangular section of the carton has a volume of 1.7 liters, which is larger than the required 1.5 L.

8. ≈ 21 cm high



Independent verification from my kitchen.

9. 19.6 yd^3

10. $53\overline{0} \text{ ft}^3$

11. 1.53 m^3

12. 327 in^3

13. $5,350 \text{ cm}^3$

14. 5.35 L

15. $0.5\ \text{ft}^3 \approx 14.16\ \text{L}$; assuming that it's $0.50\ \text{ft}^3$, these measures are equivalent to two sig figs.

16. $100\ \text{lb}/\text{ft}^3$

17. $31\ \text{lb}$; now you can laugh whenever you see someone in a heist movie load a cheap duffel bag with gold bars and carry it out of the vault.

18. $1.1\ \text{kg}$

19. $268\ \text{cm}^3$

20. $33.5\ \text{cm}^3$

21. 8 to 1

22. $1,620\ \text{in}^3$

23. $60\ \text{in}^3$

24. 27 to 1

Module 26: Pyramids and Cones

1. $1,280\ \text{cm}^3$

2. $2,420,000\ \text{m}^3$

3. $544\ \text{cm}^2$; $80\overline{0}\ \text{cm}^2$

4. $82,300\ \text{m}^2$

5. $310\ \text{cm}^3$ (if we had greater accuracy, the result would be 314.16 because it's 100 times π.)

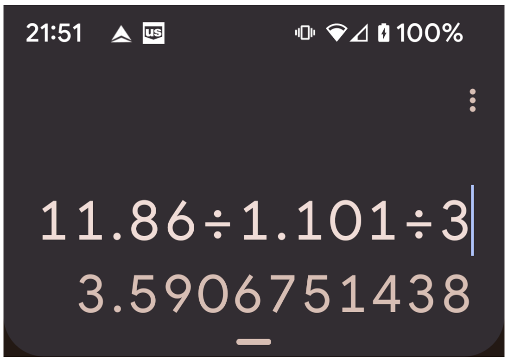
6. $38\ \text{ft}^3$

7. $47\ \text{ft}^2$; $75\ \text{ft}^2$

8. $2\overline{0}0\ \text{cm}^2$; $280\ \text{cm}^2$

Module 27: Percents Part 3

1. $\$1,299.00$
2. $14,861$ students; notice that if the percent had fewer than five sig figs, we wouldn't have been able to get an answer that was accurate to the nearest whole number.
3. Yes, you can! Each bottle cost $\$3.59$.



Real time screenshot of my phone's calculator.

4. $\$3.20$ million
5. $873,900$ people
6. $\$16$; the percent has only two sig figs, so it doesn't make sense to assume that the price was $\$16.13$. They probably rounded the percent from 68.75% to make the numbers in the advertisement seems less complicated.

Module 28: Mean, Median, Mode

1. 10.3 min
2. $\$4.488$
3. 11.0 min (because it is the seventh value in the list of thirteen)
4. $\$4.475$
5. $\$256,000$
6. $\$250,000$

7. $\$338,000$

8. $\$275,000$

9. the median is more representative because the mean is higher than five of the six home values.

10. 11.0 min (because it appears four times in the list)

11. no mode (there are no repeated values)

12. AT&T Mobility

13. Samoas and Thin Mints

14. 12.2 games

15. 12 games

16. 12 games

17. they all represent the data fairly well; 12 wins represents a typical Patriots season.

18. 6.8 games

19. 7 games

20. 6 games

21. they all represent the data fairly well; 6 or 7 wins represents a typical Bills season.

22. 98.2 grams; the mean doesn't seem to represent a typical clementine because there is a group of smaller ones (from 82 to 94 grams) and a group of larger ones (from 102 to 109 grams) with none in the middle.

23. 94 grams; for the same reason, the median doesn't represent a typical clementine, but you could say it helps split the clementines into a lighter group and a heavier group.

24. no mode; too many values appear twice.

25. 98.3 grams; this is a small increase over the previous mean.

26. 94 grams; the median does not change when one of the highest numbers increases.

27. 109 grams; you might say it represents the mass of a typical large clementine, but it doesn't represent the entire group.

Module 29: Probability

1. $\frac{6}{36} = \frac{1}{6}$

2. $\frac{2}{36} = \frac{1}{18}$

3. $\frac{8}{36} = \frac{2}{9}$

4. $\frac{36}{36} = 1$

5. $\frac{0}{36} = 0$

6. $\frac{4}{36} = \frac{1}{9}$

7. $\frac{32}{36} = \frac{8}{9}$

8. $\frac{6}{36} = \frac{1}{6}$

9. $\frac{30}{36} = \frac{5}{6}$

10. $\frac{15}{60} = \frac{1}{4} = 0.25 = 25\%$

11. $\frac{45}{60} = \frac{3}{4} = 0.75 = 75\%$

12. $\frac{18}{60} = \frac{3}{10} = 0.3 = 30\%$

13. $\frac{42}{60} = \frac{7}{10} = 0.7 = 70\%$

14. $\frac{1}{4} = 25\%$

15. $\frac{3}{4} = 75\%$

16. $\frac{2}{4} = 50\%$

17. $\frac{2}{4} = 50\%$

18. $\frac{8}{250} = 3.2\%$

19. $\frac{242}{250} = 96.8\%$

20. we should expect 968 copies to be acceptable

21. $\frac{44}{200} = 22\%$

22. we should expect 220 tax returns to have errors

23. $22\% = 0.22$

24. $(0.22)^2 \approx 4.8\%$

25. $(0.22)^3 \approx 1.1\%$

26. $0.22 \cdot 0.78 \approx 17.2\%$

27. $0.78 \cdot 0.22 \approx 17.2\%$

28. $(0.78)^2 \approx 60.8\%$

29. $(0.78)^3 \approx 47.5\%$

30. $1 - (0.78)^3 \approx 52.5\%$

Module 30: Standard Deviation

1. $60.5; 66.5$

2. $57.5; 69.5$

3. $54.5; 72.5$

4. $66.5; 72.5$

5. $63.5; 75.5$

6. $60.5; 78.5$

7. 68% because $100\% - (16\% + 16\%) = 68\%$

8. 195 lb because this is halfway between 160 and 230 lb

9. 35 lb because $195 - 35$ lb and $195 + 35$ lb encompasses 68% of the data

10. 125; 265

11. 8.8; 15.6

12. You would not have predicted this from the data because it is more than two standard deviations below the mean, so there would be a roughly 2.5% chance of this happening randomly. In fact, $(12.2 - 7) \div 1.7$ is slightly larger than 3, so this is more than three standard deviations below the mean, making it even more unlikely. (You might have predicted that the Patriots would get worse when Tom Brady left them for Tampa Bay, but you wouldn't have predicted only 7 wins based on the previous nineteen years of data.)

13. 3.4; 10.2

14. You would not predict this from the data because it is more than two standard deviations above the mean, so there would be a roughly 2.5% chance of this happening randomly. In fact, $(13 - 6.8) \div 1.7 \approx 3.6$, so this is more than three standard deviations above the mean, making it even more unlikely. This increased win total is partly due to external forces (i.e., the Patriots becoming weaker and losing two games to the Bills) but even 11 wins would have been a bold prediction, let alone 13.

15. 3.9; 14.3

16. The trouble with making predictions about the Broncos is that their standard deviation is so large. You could choose any number between 4 and 14 wins and be within the 95% interval. $(9.1 - 5) \div 2.6 \approx 1.6$, so this is around 1.6 standard deviations below the mean, which makes it not very unusual. Whereas the Patriots and Bills are more consistent, the Broncos' win totals fluctuate quite a bit and are therefore more unpredictable.

Module 31: Right Triangle Trigonometry

1. the adjacent side is *e*, the opposite side is *f*, and the hypotenuse is *d*.

2. the adjacent side is x, the opposite side is y, and the hypotenuse is r.

3. $\frac{3}{5} = 0.6$

4. $\frac{4}{5} = 0.8$

5. $\frac{3}{4} = 0.75$

6. $\frac{4}{5} = 0.8$

7. $\frac{3}{5} = 0.6$

8. $\frac{4}{3} \approx 1.333$

9. 0.6000

10. 0.8000

11. 0.7500

12. 0.8000

13. 0.6000

14. 1.333

15. $z \approx 4.6 \text{ cm}$

16. $g \approx 2.6 \text{ cm}$

17. $b \approx 5.706 \text{ in}$

18. $p \approx 75.51 \text{ mm}$

19. $y \approx 136.18 \text{ mm}$

20. $d \approx 296.87 \text{ mm}$

21. the wire is approximately 27 ft long

22. $\approx 17.85 \text{ ft}$, which is roughly $17 \text{ ft}, 10 \text{ in}$

23. No; $64 \cdot \tan 1^\circ \approx 1.1 \text{ ft}$, so the puck will hit the fabric over 1 foot away from the center of the hole. (The person in the photo was given a bunch of pucks

and 30 seconds to score, but he scored on his first shot. Boston Bruins at Vancouver Canucks, February 24, 2024.)

24. $36.87°$

25. $60°$

26. $53.13°$

27. $\angle A \approx 36.47°$

28. $\angle 1 \approx 42.03°$

29. $\angle 1 \approx 30.76°$

30. $\angle y \approx 52.88°$

31. $\angle 1 \approx 55.28°$

32. $\angle x \approx 31.50°$; $\angle y \approx 58.50°$

33. $\tan^{-1}\left(\frac{4}{1}\right) \approx 76°$ angle of elevation

34. Yes; $\sin^{-1}\left(\frac{2}{25}\right) \approx 4.59°$, which is less than $4.75°$.

35. $\tan^{-1}\left(\frac{17}{14}\right) \approx 50.5°$

36. $17 \div \sin\ 51° \approx 22.0$

37. $14 \div \cos\ 51° \approx 22.0$

38. $\sqrt{14^2 + 17^2} \approx 22.0$

39. All three answers are the same rounded to three significant figures. This is true because we rounded $\angle A$ to the nearest tenth; if we had rounded it to $51°$ instead of $50.5°$, we would have decreased the accuracy of #36 & #37 to only two sig figs and the three results all would have been slightly different.

40. $\approx 23°$

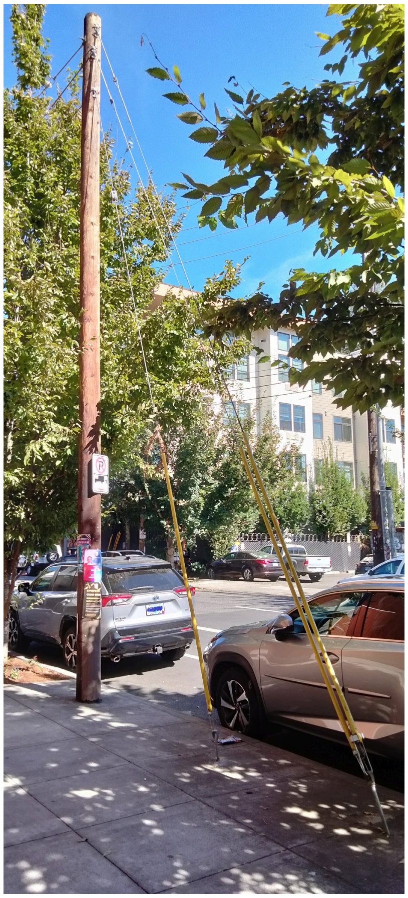
41. $\approx 8,900\ \text{ft}$

Module 32: Slope

1. 0.36:12, 0.03, 3%
2. 0.60:12, 0.05, 5%
3. 1.00:12, 0.0833, 8.33%
4. 2.16:12, 0.18, 18%
5. $1.72°$
6. $2.86°$
7. $4.76°$
8. $10.20°$
9. 0.5 ft
10. 1 ft
11. 5 ft
12. $\tan^{-1}(20/12) \approx 59°$ from vertical. (this would be an angle of depression of $31°$.)
13. 30 ft; the proportion $\frac{1}{12} = \frac{2.5}{x}$ gives a result of exactly 30 ft.
    Using the result from #7, the equation $\tan(4.76°) = \frac{2.5}{x}$ gives a result very close to 30.0 ft.

Module 33: Non-Right Triangle Trigonometry

1. $k \approx 61 \text{ in}$
2. $a = c = 74°$; $x \approx 5.168 \text{ in}$
3. $x \approx 11.6 \text{ cm}$; $y \approx 12.3 \text{ cm}$
4. the shorter wire is roughly 23.5 feet; the longer wire is roughly 26 feet.



Exercise 4 is based on actual events!

5. $n \approx 72.4°$

6. $n \approx 107.6°$

7. $d \approx 65$ mm

8. $p \approx 11.4$ cm

9. $w \approx 14.12$ m

10. the distance across the pond is roughly 81.5 meters.

11. $X \approx 19.4°$; $Y \approx 133.5°$; $Z \approx 27.1°$

Module 34: Radian Measure

1. $\frac{\pi}{6}$

2. $\frac{\pi}{4}$

3. $\frac{\pi}{3}$

4. $\frac{\pi}{2}$

5. $\frac{2\pi}{3}$

6. $\frac{5\pi}{4}$

7. $\frac{4\pi}{3}$

8. $\frac{3\pi}{2}$

9. $15°$

10. $135°$

11. $150°$

12. $315°$

13. $\approx 57.3°$